SOME PROBLEMS IN KNOT THEORY

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ABSTRACT

Some problems in knot theory

Following the work of Lickorish and Millett on the two variable polynomial, $P(\ell,m)$, generalizations and geometric interpretations of the polynomial coefficients of all powers of m are investigated using the substitution $\ell = 1$. Corollaries regarding the reversal of orientations on the components of 2- and 3-component links and the effect on coefficients of the Conway polynomial are given as well as general reversing results for the Arf invariant of links.

The first $c - 1$ polynomial coefficients of $P(\ell,m)$ are shown to encode various products of linking numbers which generalize a result of Hoste's on the evaluation of the (normalized) Conway polynomial, $\tilde{\nabla}_A(z)$, at $z = 0$.

Problems associated with chirality are studied via the braid groups. Using self-maps of B_n, non-trivial achiral closed braids are constructed. Braids constructed in this way are called visibly achiral and visibly reversed achiral. It is shown that a 3-braid is conjugate to its mirror image if and only if the braid is visibly achiral. In fact, as a converse to a conjecture of Kauffman's, it is shown that a necessary condition for knot represented by an alternating closed braid to have its graph isomorphic to its dual is that the braid be visibly achiral.

A new invariant of braid conjugacy class is presented and is used to show an example of an achiral alternating closed braid diagram on 5 strings that is not conjugate to its mirror image. It is then not visibly achiral and is hence a counter-example to Kauffman's conjecture.

ACKNOWLEDGEMENTS

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PREFACE

> Knot theory was born around the year 1867 in Scotland from the imagination of three physicists; two Scotsmen living in Edinburgh: J. C. Maxwell and P. G. Tait and one Irishman living in Glasgow: W. Thomson (Lord Kelvin).
>
> *De La Harpe, Kervaire and Weber, 1986* [HKW, p.271].

In the past several years there has been a resurgence of the old view of using projections of knots and links and it is from such a framework of diagrams that this paper is based.

The following text is divided into two parts with chapters 1 and 3 serving to introduce the notation, definitions and basic ideas necessary for the chapters that succeed them.

Chapter 2 deals with the Conway polynomial, $\nabla(z)$, and the two-variable generalization of the Jones polynomial, $P(\ell,m)$, more precisly, with the invariants the evaluation at $\ell = 1$ yield in the Laurent polynomial coefficients of m^k. For a c-component link, the substitution of $\ell = 1$ into the polynomial $P(\ell,m)$ yields the Conway poly nomial with the first $c - 1$ terms going to 0. After factoring out obvious zeros these first $c - 1$ terms then become non-trivial invariants of link type. A formula for their calculation is given in terms of products of linking numbers, generalizing an earlier result of Hoste's.

The total twisting of a c-component link, τ_c, is defined and shown to be (up to the sign) the coefficient of z^{c+1} in $\nabla_o(z)$. An ordered list of the crossings of a knot diagram is used in an alternative definition of total twisting of a knot giving a much easier method of calculation. Likewise, τ_2 is computed in an analogous way. An example of the calculation is given which also demonstrates that τ_2 can detect the bond between two components with linking number zero.

Propositions 2.1.11 and 2.2.5 along with Appendix A are concerned with the reversing of orientations on components and its effect on τ, τ_2 and τ_3, respectively. 2.1.11 Relates the total twisting of knots obtained by nullifying corresponding crossings of two 2-component link diagrams that differ by an orientation on one of the components. A corollary to this is a formula for τ of a $(2,2q-1)$ cable knot based on a companion. Using 2.2.5 a similar corollary is reached dealing with the $(2,2q)$ cable link based on a companion.

Chapters 3 and 4 are concerned with braids. A new invariant Φ is defined and an example is given where Φ detects that two braids with isotopic closures are non-conjugate. The writhe (which plays an important role in the new polynomials) is investigated for ascending braids in order to obtain some corollaries. One of these is used in chapter 4 to show that the polynomial $P(\ell,m)$ is symmetric in ℓ, $-\ell^{-1}$ for any braid on 3 strings with zero writhe. This produces many examples where the symmetry property holds but the closed braid is chiral.

One way of obtaining achiral links is by exploiting certain flipping and reflecting operations within the rich structure of the braid groups. The braids so obtained are termed visibly achiral. It is shown that a 3-braid is conjugate to its mirror image if and only

if it takes a certain symmetric form. Corollaries to this prove that if a 3-braid is conjugate to its mirror image then the closure is an alternating link, moreover, that any alternating closed braid diagram of such a braid must be in this symmetric form. Analogous results are established for braids conjugate to the reverse of their mirror image. The analysis has lead to a closed braid on 5 strings that is a counter-example to a conjecture by Kauffman. Finally, a converse to Kauffman's conjecture for all n-stringed closed braids is proven.

CONTENTS

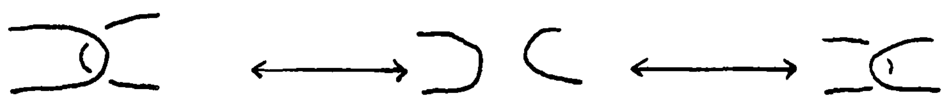
CHAPTER 1

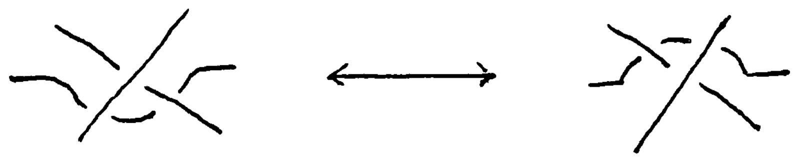
1.1 Notation and definitions

A link consists of a finite disjoint union of oriented circles tamely embedded in 3-space. A link universe (or shadow) of a link is the image of the link under a projection to the plane, where the projection is such that there are at most finitely many self-intersection points and each such point is a transverse double point. A link diagram consists of a link universe together with an assignment of the overcrossing and undercrossing at each double point called a crossing. For a link universe with n crossings there are 2^n possible assignments, i.e. 2^n possible link diagrams with that shadow. Two link diagrams A and B will be regarded as the same, denoted $A = B$ if they are identical up to planar isotopy (motions of the diagrams that preserve the graphical strcture of the underlying universe as well as the assignment of over and undercrossings, (see [K3,p.3])). Every link diagram determines a link, up to isotopy, in the obvious way. Conversely, every suitable projection of a link determines a link diagram, which we say represents the link, and two links represented by the same diagram are isotopic.

Two link diagrams may yield the same link, up to isotopy. Indeed [Re], they do so if one can be obtained from the other by a sequence of Reidemeister moves shown below:

R1. 

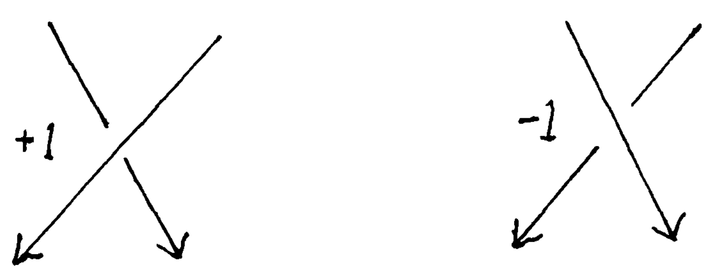
R2. 

R3. 

Moreover, if K and L are links represented by diagrams A and B, repectively, such that K is isotopic to L, denoted $K \cong L$, then A can be obtained from B by a sequence of Reidemeister moves. If $K \cong L$ then we say A is isotopic to B and write $A \cong B$.

The number of components of a link L is denoted c(L) or c(A) where A is a diagram representing L. In particular, if c(L) = 1 then we say L is a <u>knot</u> and A a <u>knot diagram</u>. Write U^c for the link diagram of c components with no crossings. Say the link U^c represents the <u>trivial link</u>.

For each crossing of an oriented link diagram the assignment of over- and under-crossing is called the sign of the crossing and is denoted by $\varepsilon = \pm 1 \in \mathbb{Z}$ according to the following:



Let A be a link diagram with n crossings $d_1, d_2, \ldots, d_n$ with signs $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. Define the <u>writhe</u> of a link diagram A, $\omega(A)$ to be the sum of the signs over all crossings;

$$\omega(A) = \sum_{i=1}^{n} \varepsilon_i$$

If A has $c(A) = k$ components, then we can label the components $C_1, C_2, \ldots, C_k$. For a crossing d_h of components C_i and C_j call d_h a knot crossing of A if $i = j$ and a link crossing of A otherwise. Define the linking of components C_i and C_j, denoted $\lambda(C_i, C_j)$, to be one half the sum of the signs over all crossings of components C_i and C_j, $i \neq j$. Likewise, define the total linking of A, denoted $\lambda(A)$, as follows:

$$\lambda(A) = \tfrac{1}{2} \Sigma \varepsilon_i$$

where d_i is a link crossing of A. Notice that the linking and total linking are always integer valued.

For a diagram A define the self-writhe of a component C_i to be the sum of the signs over all crossings of C_i with itself and define the self-writhe of A, $sw(A)$, as follows;

$$sw(A) = \Sigma \varepsilon_i$$

where d_i is a knot crossing of A. We then have

$$\omega(A) = 2\lambda(A) + sw(A) \tag{1.1.1}$$

If A is a link diagram with crossings $d_1, d_2, \ldots, d_n$ and signs $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ then define the obverse of A to be the diagram $\bar{A}$ with crossings $d_1, d_2, \ldots, d_n$ but with signs $-\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_n$, where A and $\bar{A}$ have the same shadow and orientation. The link $\bar{L}$ represented by the link diagram $\bar{A}$ is called the mirror image of the link L represented by A. If A and $\bar{A}$ represent isotopic links then L is said to be achiral. If $\tilde{A}$ is the diagram A with its orientation reversed then L is said to be reversed achiral if $\tilde{A}$ and $\bar{A}$ represent isotopic links. L is unorientedly achiral if L is either achiral or reversed achiral and is otherwise chiral.

Two easy methods of constructing achiral knots and links are by taking the connected sum of a diagram A and its obverse, denoted by

$A \# \bar{A}$, and by taking the separated union, denoted by $A \sqcup \bar{A}$.

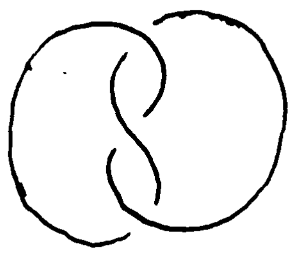


Diagram A

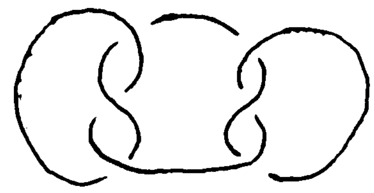


Diagram $A \# \bar{A}$

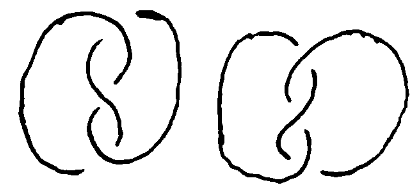
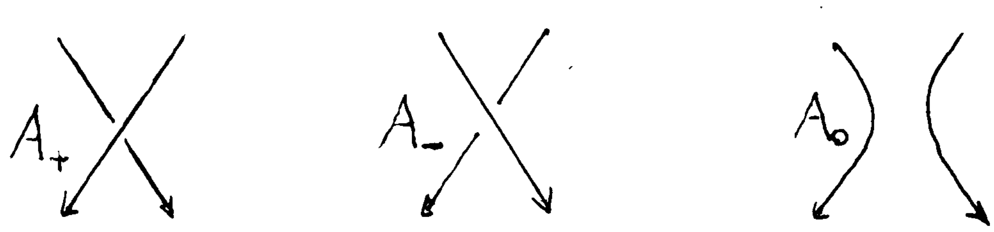


Diagram $A \sqcup \bar{A}$

1.2 Link polynomials

The work of Conway [**Co**] and Jones [**J1**] has inspired the construction of various new polynomials defined recursively using relations between diagrams of different "complexity". Roughly speaking, "complexity" is measured by the number of crossings in a link diagram and by the length of a sequence of crossing switches that yield the trivial link. Suppose the oriented diagrams A_+, A_- and A_0 are exactly the same except near one point where they differ as in the figure below.



Then A_0 is "less complex" than either A_+ or A_-, and (for suitable choice of crossing) one of A_+ or A_- is "less complex" than the other. Any such triple of diagrams will be referred to as Conway diagrams.

Every link diagram has at least one sequence of crossing switches that yield a trivial link diagram. There is a canonical method of obtaining a trivial link diagram by switching over-crossings to under-crossings and vice versa and the one described here is from [**LM1**]: For a link diagram A, choose a base point p_i for each of the ordered components $C_1, C_2, \ldots, C_k$ of A (p_i chosen not to be a crossing). Switch every link crossing so that C_i passes over C_j for $1 \leqslant i < j \leqslant k$. Then starting from p_i for each $i = 1,2,\ldots,k$, traverse C_i switching (where necessary) every knot crossing so that it is first encountered as an undercrossing. Define the resulting diagram to be <u>ascending</u>. If B is an ascending diagram with $c(B) = k$ components then $B \cong \mathcal{U}^k$.

Conway, in a paper released in 1969, noticed that the classical normalized Alexander polynomial invariant of a link represented by a diagram A, denoted $\Delta_A(t)$ satisfies

$$\Delta_{A_+}(t) - \Delta_{A_-}(t) + (t^{½} - t^{-½})\, \Delta_{A_0}(t) = 0$$

where $t^{½}$ is the formal square root of t. Hence, the Alexander polynomial could be defined and evaluated recursively. In [**J1**], Jones found (via Von Neumann algebras and their relation to the braid groups) a new polynomial invariant $V(t)$ that satisfies

$$t\, V_{A_+}(t) - t^{-1}\, V_{A_-}(t) + (t^{½} - t^{-½})\, V_{A_0}(t) = 0$$

Nowadays, there is a concrete geometrical interpretation of the (Hecke) algebras in terms of a diagram monoid, D_n, generated by elements called "hooks" that yield a representation of the braid groups. A trace funtion on D_n is then used to define the Jones polynomial [**K3**].

Almost immediately after Jones' work became known, at least four other groups [**HOMFLY**], working independently generalized his polynomial to one on two variables that also gave the Alexander, Conway and Jones polynomials as particular instances.

1.3 The H.O.M.F.L.Y. polynomial

1.3.1 Definition: The H.O.M.F.L.Y. polynomial of a diagram A will be denoted $P(A)(\ell,m) \in \mathbb{Z}[\ell^{\pm 1},m^{\pm 1}]$ or simply $P(A)$. It satisfies the following axioms;

(i) If a diagram B is such that $A \cong B$, then

$$P(A)(\ell,m) = P(B)(\ell,m)$$

(ii) $P(U)(\ell,m) = 1$

(iii) The fundamental relation;

$$\ell\, P(A_+)(\ell,m) - \ell^{-1}\, P(A_-)(\ell,m) + m\, P(A_0)(\ell,m) = 0,$$

where A_+, A_- and A_0 are the Conway diagrams as previously defined.

When there is a need to be more specific about the point where the Conway diagams differ the notation will be as follows. Let A be a link diagram with crossings $d_1, d_2, \ldots, d_n$, each with sign $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. Let $\xi_i A$ denote the switching of d_i in A from sign ε_i to sign $-\varepsilon_i$ and $\eta_i A$ the nullifying of d_i, i.e., if $A_{\varepsilon_i} = A$ then $A_{-\varepsilon_i} = \xi_i A$ and $A_0 = \eta_i A$. One notes that $\xi_i \xi_j A = \xi_j \xi_i A$, $\eta_i \eta_j A = \eta_j \eta_i A$ and $\xi_i \eta_j A = \eta_j \xi_i A$ for all crossings d_i and d_j.

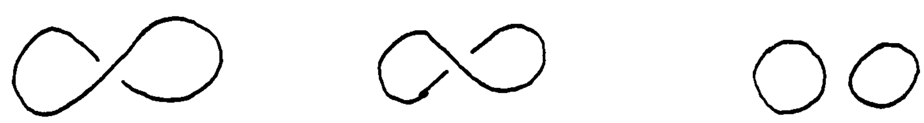
Because of (i) above (for a proof of which the reader is referred to [**HOMFLY**]) we often write $P(A)(\ell,m) = P(L)(\ell,m)$ or simply $P(A) = P(L)$ where the diagram A represents the link L.

The following properties of $P(A)$ will be needed and we will refer back to them frequently. All but P7. can again be found in [**LM1**] and the statements following each are intended more as exemplification of the ideas than as rigorous proofs.

<u>Proprties of $P(A)(\ell,m)$</u>

<u>P1</u>. $P(U^c) = \mu^{c-1}$,where $\mu = - m^{-1}(\ell - \ell^{-1})$.

<u>Sketch</u>: Let A_+, A_- and A_0 be as follows.



A_+ Positive Kink $\quad$ A_- Negative Kink $\quad$ A_0

Then the fundamental relation yields

$$P(A_0) = - m^{-1}[\ell\ P(A_+) - \ell^{-1}\ P(A_-)]$$

and by axiom (ii) A_+ and A_- , being diagrams of the trivial knot, have polynomials $P(A_+) = P(A_-) = 1$ and the result for $c = 2$ follows. Similarly, by kinking one of the components of U^c and inducting on c one obtains the desired result. □

<u>P2.</u> The lowest power of m in $P(L)(\ell,m)$ is $1 - c$, where $c(L) = c$ and the powers of ℓ and m are either all even or all odd accordingly as $1 - c$ is even or odd.

<u>Sketch</u>: Here we use induction on the "complexity" of the diagrams A_+, A_- and A_0 and assume that the property is true for A_- and A_0. Notice that $c(A_+) = c(A_-) = c(A_0) \pm 1$. The fundamental relation gives $P(A_+) = \ell^{-2}\ P(A_-) - \ell^{-1}\ m\ P(A_0)$. By the induction, the monomial $\ell^{-2}\ P(A_-)$ in $P(A)$ clearly satisfies P2. and because

$1 - c(A_0)$ is at most c and congruent to c (modulo 2), the monomial $- \ell^{-1} m\, P(A_0)$ in $P(A)$ also satisfies P2.. □

<u>P3.</u> Reversing the orientation of every component of L leaves the polynomial unchanged.

Here we need only note that reversing the orientation of every component of L does not affect either the sign of any crossing nor the "complexity" of the diagrams A_+, A_- and A_0. If the orientation of some of the components is reversed the polynomial will, in general, change. Some consequences of this will be looked at in chapter 2.

<u>P4.</u> If $\bar{A}$ is the obverse of a diagram A then

$$P(A)(\ell,m) = P(\bar{A})(-\ell^{-1},m).$$

Note that for every sequence of crossing switches that yield a trivial link for A there is an analogous sequence in $\bar{A}$; the role of A_+ and A_- are simply switched in each set of Conway diagrams.

<u>P5.</u> $P(A \sqcup U) = \mu\, P(A)$ and hence for any diagram B, $P(A \sqcup B) = \mu\, P(A)\, P(B)$, where $\mu = m^{-1}(-\ell + \ell^{-1})$.

<u>Sketch</u>: Assume inductively that Conway diagrams A_- and A_0 are "less complex" than A_+. Then

$$\begin{aligned} P(A_+ \sqcup B) &= \ell^{-2}\, P(A_- \sqcup B) - \ell^{-1}\, m\, P(A_0 \sqcup B) \\ &= [\ell^{-2}\, P(A_-) - \ell^{-1}\, m\, P(A_0)]\, P(B)\, \mu \\ &= P(A)\, P(B)\, \mu . \end{aligned}$$

□

<u>P6.</u> $P(A \# B) = P(A)\,P(B)$ regardless of how the oriented union is formed.

<u>Sketch</u>: Analogous to the sketch proof of P5. □

<u>P7.</u> **[Mo3]** Let A be a link diagram with crossings $d_1, d_2, \ldots, d_n$ with signs $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. Define $s(A)$ to be the number of components in the diagram $\prod_{i=1}^{n} \eta_i A$ (i.e. the number of Seifert circles). Let e and E denote the minimum and maximum powers of ℓ respectively and M to be the maximum power of m in $P(A)(\ell,m)$, then

$$\text{(i)}\quad -\omega(A) - [s(A) - 1] \leqslant e \leqslant E \leqslant -\omega(A) + [s(A) - 1]$$

$$\text{(ii)}\quad M \leqslant n - [s(A) - 1].$$

<u>Sketch</u>: (i) Using the relation between A and its obverse, one need only show that $\varphi(A) \leqslant e$ where $\varphi(A) = -\omega(A) - [s(A) - 1]$, or equivalently that $\ell^{-\varphi(A)} P(A)(\ell,m)$ has no negative powers of ℓ. Use induction on the number of crossings and note that if A_+, A_- and A_0 are as previously defined then $\varphi(A_+) + 1 = \varphi(A_0) = \varphi(A_-) - 1$. Hence, by the fundamental relation

$$\begin{aligned} 0 &= \ell^{-\varphi(A_0)}[\ell\, P(A_+) - \ell^{-1}\, P(A_-) + m\, P(A_0)] \\ &= \ell^{-\varphi(A_+)}\, P(A_+) - \ell^{-\varphi(A_-)}\, P(A_-) + \ell^{-\varphi(A_0)}\, m\, P(A_0) \end{aligned}$$

so that by the induction $\ell^{-\varphi(A_+)}\, P(A_+)$ is a polymomial with no negative powers of ℓ if and only if $\ell^{-\varphi(A_-)}\, P(A_-)$ is. From this relation, one has that if A' is an ascending diagram obtained from

A then $\ell^{-\varphi(A)}$ P(A) has no negative powers of ℓ if and only if $\ell^{-\varphi(A')}$ P(A') has none. Noting that the lowest power of ℓ in P(A') is $1 - c(A)$ it remains to show that $-\varphi(A') \geqslant c - 1$. This, however, is true independently of the polynomial and (i) follows.

(ii) This is almost immediate from the definition of s(A) given above. □

1.4 The Conway Polynomial

The following chapter looks at specific coefficients of the Conway polynomial. The purpose of this section is to obtain some of the basic results that will be used throughout the chapter.

The Conway polynomial, $\nabla_A(z)$, of a link diagram A may be obtained from $P(A)(\ell,m)$ using the substitution $\ell = 1$ and $m = z$. We may then adapt and summarize P2 and P5 as follows:

C1. $\tilde{\nabla}_A(z) = \nabla_A(z)/z^{c(A)-1}$ is a polynomial in z^2.

C2. $\nabla_{A \sqcup B}(z) = 0$, where $A \sqcup B$ denotes the separated union of diagrams A and B.

Using C1, we write $\tilde{\nabla}_A(z)$, the normalized Conway polynomial, in canonical form as follows:

$$(1.4.1) \qquad \tilde{\nabla}_A(z) = \sum_{i=0}^{N} a_{2i} \cdot z^{2i}$$

for some $N \in \mathbb{Z}$.

Nearly all proofs in the next chapter use an induction on a sequence of crossing switches that take a "complicated" link to a "simpler" one. Let A be a c-component link diagram and $\xi_1, \xi_2, \ldots, \xi_s$ a sequence of crossing switches that change the sign of crossings d_i from ε_i to $-\varepsilon_i$ resulting in a diagram B. Then, repeated use of the defining relation gives

$$(1.4.2) \qquad \nabla_A(z) = \nabla_B(z) - z \sum_{j=1}^{s} \varepsilon_j \cdot \nabla_{A_j}(z)$$

where $A_j = (\eta_j \prod_{i<j} \xi_i)A$.

<u>1.4.3 Proposition</u>:

(1) If A is a knot diagram then $\tilde{\nabla}_A(0)$ $[= \nabla_A(0)] = 1$

(2) If A is a link diagram then $\tilde{\nabla}_A(0) = -\lambda(A)$

<u>Proof</u>: Notice that for a knot crossing, d_i, of a link diagram A, $c(\eta_i A) = c(A) + 1$ and that for a link crossing $c(\eta_i A) = c(A) - 1$. This implies that in the normalized Conway polynomial, $\tilde{\nabla}_A(z)$, if $\xi_1, \xi_2, \ldots, \xi_s$ is a sequence of knot crossings that unknot A then

using (1.4.2) we have

$$(1.4.4) \qquad \tilde{\nabla}_A(z) = \tilde{\nabla}_B(z) - z^2 \sum_{j=1}^{s} \varepsilon_j \; \tilde{\nabla}_{A_j}(z)$$

$\tilde{\nabla}_B(z) = 1$ by definition. Hence, putting zero in for z completes the proof of (1). Likewise, if $\xi_1, \xi_2, \ldots, \xi_s$ is a sequence of crossing switches that unlink the components of A then

$$(1.4.5) \qquad \tilde{\nabla}_A(z) = \tilde{\nabla}_B(z) - \sum_{j=1}^{s} \varepsilon_j \cdot \tilde{\nabla}_{Aj}(z).$$

Here, $\tilde{\nabla}_B(z) = 0$ by C2 and for each j, $\tilde{\nabla}_{A_j}(0) = 1$ by (1).

Hence, $\tilde{\nabla}_A(0) = -\sum\limits_{j=1}^{s} \varepsilon_j = -\lambda(A)$. □

CHAPTER 2

2.1 The total twisting, $\tau(K)$

In [LM1], Lickorish and Millett define the total twisting, $\tau(A)$, of an oriented knot diagram A and show that it can be calculated from $P(A)(\ell,m)$, thereby proving it is an invariant of knot type. This calculation is equivalent to finding the second coefficient of the normalized Conway polynomial, $\tilde{\nabla}_A(z)$. The invariance of $\tau(A)$ will be proved here using this relation to the Conway polynomial instead of $P(A)(\ell,m)$. We now define $\tau(A)$ as in [LM1].

2.1.1 Definition: Let A be a diagram which represents a knot K. Given a sequence of crossing switches, ξ_j, $j = 1,2,\dots,r$ of crossings of sign ε_j, which yield a diagram of the trivial knot, there is an associated sequence of two component link diagrams

$$A_j = (\eta_j \prod_{i<j} \xi_i)A = B_{j_1} \cup B_{j_2}$$

from which one defines the total twisting

$$\tau(A) = \Sigma\, \varepsilon_j \cdot \lambda(B_{j_1}, B_{j_2}).$$

2.1.2 Proposition: Let A be a diagram which represents a knot K. Let $a_2(K)$ be the coefficient of z^2 in $\nabla_A(z)$. Then

$$a_2(K) = \tau(K).$$

Proof: By (1.4.2) we may write

$$\tilde{\nabla}_A(z) = 1 - z^2 \sum_{j=1}^{s} \tilde{\nabla}_{A_j}(z) \cdot \varepsilon_i$$

Hence

$$a_2(A) = -\Sigma\, a_o(Aj)\cdot\varepsilon_j$$

$$= \Sigma\, \lambda(A)\cdot\varepsilon_j = \tau(A). \qquad \square$$

Let A be a knot diagram. Label the crossings $d_1, d_2, \ldots, d_n$ and let ε_i be the sign of d_i. Let A' be a knot diagram obtained from A by switching all the crossings in $D = \{d_{j_1}, d_{j_2}, \ldots, d_{j_r}\}$. Then the following defines a useful generalization of $\tau(A)$.

<u>2.1.3 Definition</u>: Let A and A' be as above, then define $\tau(A:A')$ as follows.

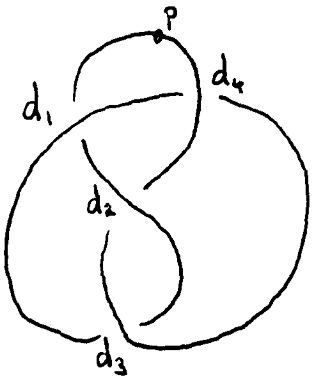
(i) Starting at a base point p (not on a crossing) traverse A in the direction of the orientation. List all crossings in the order in which they occur. [Note that in one trip "around" A each crossing will be met twice and should appear twice in the list]. Define this to be the <u>list of A</u> and denote it by List(A).

(ii) Consider pairs (j,k) where $d_j \in D$ and $d_k \notin D$ with d_j occurring exactly once in between the two d_k in List(A).

(iii) Then $2\cdot\tau(A:A') = \Sigma\, \varepsilon_j\cdot\varepsilon_k$ where the sum is taken over all pairs (j,k) as defined in (ii).

2.1.4 Example:

Let A be a diagram of the figure-of-eight knot. A has crossings $d_1, \ldots, d_4$, with signs $\varepsilon_1 = \varepsilon_4 = -1$ and $\varepsilon_2 = \varepsilon_3 = 1$.

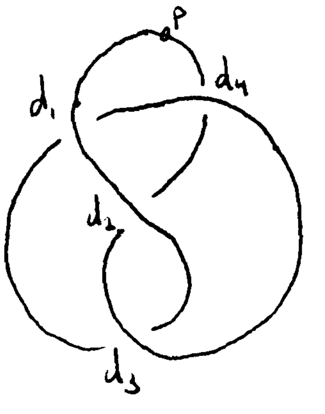


A

Switching the crossings in $D = \{d_1, d_4\}$ yields a diagram, A' , of the trefoil knot. $\text{List}(A) = d_1, d_2, d_3, d_1, d_4, d_3, d_2, d_4$.

Then $2\,\tau(A:A') = \varepsilon_1(\varepsilon_2 + \varepsilon_3) + \varepsilon_4(\varepsilon_2 + \varepsilon_3)$

$= -2 - 2 = -4.$



A'

Hence, $\tau(A:A') = -2$.

The invariance of $\tau(A:A')$ is due to the invariance of $\tau(A)$ and $\tau(A')$. We have

2.1.5 Proposition: Let A, and A' be any two diagrams with the same shadow. Then

$$\tau(A) = \tau(A:A') + \tau(A').$$

Proof: Let ξ_j, $j = 1, 2, \ldots, r$ be a sequence of crossing changes that takes A to A'. Clearly, we have that

$$\tau(A) = \sum_{j=1}^{r} \varepsilon_j \cdot \lambda(A_j) + \tau(A')$$

where $A_j = (\eta_j \prod_{i<j} \xi_i)A$. We need only prove, then, that

$$\tau(A:A') = \sum_{j=1}^{r} \varepsilon_j \cdot \lambda(A_j).$$

For any crossing d_j the link crossings of the diagram $\eta_j A$ correspond to those in the List(A) that appear exactly once in between the two d_j. Hence, for each j we have that $\varepsilon_j \cdot \lambda(\eta_j A) = 1/2 \ \Sigma\ \varepsilon_j \cdot \varepsilon_k$; where d_k appears once in between the two d_j in the list List(A). This also implies that d_j is a link crossing of $\eta_h A$ if and only if d_h is a link crossing of $\eta_j A$, so that if both $d_h, d_j \in D$ with $j < h$ then the effect $\varepsilon_j \cdot \varepsilon_h$ has in $\varepsilon_j \cdot \lambda(A_j)$ is cancelled by the effect $(\varepsilon_h)(-\varepsilon_j)$ has in $\varepsilon_h \cdot \lambda(A_h)$ when summing over all $\varepsilon_k \cdot \lambda(A_k)$ as k goes from 1 to r. Therefore, as the definition of $\tau(A:A')$ states, we need only consider products $\varepsilon_j \cdot \varepsilon_h$ where one of d_j or d_h is in D. □

Hence $\tau(A:A')$ is invariant under the knot types of A and A'. An interesting feature of $\tau(A:A')$ for general A and A' is its additive nature.

<u>2.1.6 Proposition</u>: For any three knot diagrams A, A' and A" with the same shadow

$$\tau(A:A") = \tau(A:A') + \tau(A':A").$$

<u>Proof</u>: The proof is immediate from 2.1.5. □

Some corollaries to 2.1.6 are easily derived with various choices for the diagrams A, A' and A". Unless defined otherwise we take these diagrams to be any three that share the same shadow.

<u>2.1.7 Corollary</u>: $\tau(A:A') = -\tau(A':A)$.

<u>Proof</u>: Clearly $\tau(A:A) = 0$ so that by the proposition
$0 = \tau(A:A) = \tau(A:A') + \tau(A':A)$. □

<u>2.1.8 Corollary</u>: $\tau(A:\bar{A}) = 0$, where $\bar{A}$ is the obverse of A.

<u>Proof</u>: Note first that if A'' is any unorientedly achiral knot with the same shadow as A then A'' and $\bar{A}''$ represent the same unoriented knot and so

$$\begin{aligned}\tau(A:\bar{A}) &= \tau(A:A'') - \tau(\bar{A}:A'') \\ &= \tau(A:A'') - \tau(\bar{A}:\bar{A}'')\end{aligned}$$

using the invariance of $\tau(\bar{A}:A'')$. Observe that for each term $\varepsilon_i \cdot \varepsilon_j$ in the calculation of $\tau(A:A'')$ we have the corresponding term $(-\varepsilon_i)(-\varepsilon_j)$ in the calculation of $(\bar{A}:\bar{A}'')$, hence the two must be equal. □

<u>2.1.9 Corollary</u>: Let A, A' be any two diagrams with ascending diagrams B, B', respectively. Then

$$\tau(A \# A':B \# B') = \tau(A:B) + \tau(A':B')$$

or equivalently

$$\tau(A \# A') = \tau(A) + \tau(A')$$

<u>Proof</u>: In the List($A \# A'$), no crossing of A appears in between any two occurrences of a crossing of A' so that
$\tau(A \# A':A \# B') = \tau(A':B')$ and $\tau(A \# B':B \# B') = \tau(A:B)$.

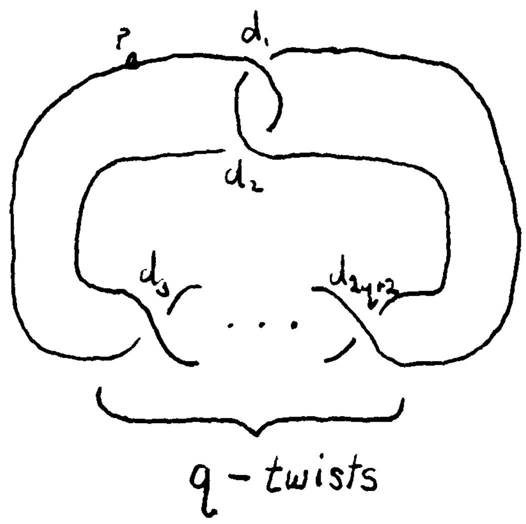
We have then that

$$\tau(A \# A':B \# B') = \tau(A \# A':A \# B') + \tau(A \# B':B \# B')$$

and the result follows. □

<u>2.1.10 Corollary</u> (see [**LM1**]): Let A be the diagram of the twist knot with q twists, $q \in \mathbb{Z}$ and A' a diagram of the unknot with the same shadow, then $\tau(A:A') = \tau(A) = -q$.

<u>Proof</u>: We have crossings $d_1, d_2, \ldots, d_{2q+2}$ with signs $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_i = -1$ if $q>0$, $\varepsilon_i = +1$ otherwise, and that $D = \{d_1\}$ (as shown).



$\text{List}(A) = d_1, d_2, \ldots, d_{2q+2}, d_2,$

$\qquad d_1, d_{2q+2}, d_{2q+1}, \ldots, d_3$

Hence, $\tau(A) = 1/2 \sum_{i=3}^{2q+2} \varepsilon_1 \cdot \varepsilon_i = -q.$ □

The following proposition will be the foundation of the theory to follow, and, as corollary 2.1.14 demonstrates, it can sometimes be useful in its own right. We set aside $\tau(A:A')$ until the next section and now concentrate on $\tau(A)$ as defined in [**LM1**].

Let A be a 2-component link diagram in which the components separately form knot diagrams B_1 and B_2. Denote by $\tilde{A}$ the 2-component link diagram A with its orientation reversed on exactly one of its components. Let $\lambda = \lambda(A)$. Note that $\lambda(A) = -\lambda(\tilde{A})$ and that $sw(A) = sw(\tilde{A})$. Let d be a link crossing of A of sign ε

then d is also a link crossing of $\tilde{A}$, but of sign $-\varepsilon$. Let $A_0 = \eta A$ and $A_\infty = \eta\tilde{A}$. Then we have

2.1.11 Proposition: Given the above conditions

$$\tau(A_0) = -\tau(A_\infty) + 2\,[\tau(B_1) + \tau(B_2)] + 1/2\cdot\lambda\cdot(\lambda - \varepsilon).$$

Proof: To set up an induction on the length r, of a sequence of crossing switches that yield the separated union of components B_1 and B_2 (up to isotopy), suppose first that $r = 0$, that is, all the link crossings of A have B_1 pass over (say) B_2 and so $\lambda = 0$. Nullifying any link crossing yields both A_0 and A_∞ as connected sums and hence, by 2.1.9

$$\tau(A_0) = -\tau(A_\infty) + 2\,[\tau(B_1) + \tau(B_2)]$$

Now suppose $r = 1$, i.e. switching the link crossing d (say) gives (up to isotopy) the separated union of B_1 and B_2. Then nullifying d also yields a connected sum with $\lambda - \varepsilon = 0$.

Assume inductively that the proposition has been established for all sequence lengths less than r and that $d_1, d_2, \ldots, d_r$ is a sequence of switches from sign ε_i to $-\varepsilon_i$ which gives the separated union of B_1 and B_2. Nullifying $d = d_1$ (say) in both A and $\tilde{A}$ yields A_0 and A_∞ respectively. Operating on d_2 (say) in A_0 we obtain from the definition of $\tau(A_0)$

$$\tau(A_0) = \tau(\xi_2 A_0) + \varepsilon_2\cdot\lambda(\eta_2 A_0)$$

By the induction we get

$$\tau(A_0) = -\tau(\xi_2 A_\infty) + 2\cdot[\tau(B_1) + \tau(B_2)] + 1/2\cdot(\lambda - \varepsilon_2)(\lambda - \varepsilon_2 - \varepsilon_1)$$
$$+ \varepsilon_2\cdot\lambda(\eta_2 A_0)$$

Notice that $\tau(A_\infty) = \tau(\xi_2 A_\infty) + (-\varepsilon_2)\cdot\lambda(\eta_2 A_\infty)$

We now interrupt the proof to establish

2.1.12 Lemma: Given the above conditions

$$\lambda(\eta_2 A_0) = \lambda(\eta_2 A_\infty) + \lambda - 1/2\cdot(\varepsilon_1 + \varepsilon_2)$$

Proof: Consider the components B_1 and B_2 in A. Let $d_{r+1},\ldots,d_n$ be the remaining $n - r$ crossings of A with ε_i the sign of d_i. Choose base points p_j of B_j for $j = 1$ or 2. Starting at p_j, traverse B_j in the direction of its orientation listing all crossings in the order in which they occur. Define this to be the sublist of B_j denoted by Sublist(B_j). Note that link crossings appear once in each sublist; knot crossings, twice. Also that List(B_j) = Sublist(B_j) if and only if A is the distant union of components B_1 and B_2.

Now choose p_j such that d_1 appears first in both sublists. Let $S_{1,j} = \{d_k\colon k \neq 1$ and d_k appears before d_2 in Sublist$(B_j)\}$ and $S_{2,j} = \{d_k\colon k \neq 1$ and d_k appears after d_2 in Sublist$(B_j)\}$. Notice that $S_{1,1} \cup S_{2,2}$ are precisely the crossings on one of the components of $\eta_2 A_0$ and $S_{1,2} \cup S_{2,1}$ are the crossings on the other component. Hence $\lambda(\eta_2 A_0) = 1/2\cdot\Sigma\, \varepsilon_i$ where d_i is in $\{(S_{1,1} \cup S_{2,2}) \cap (S_{1,2} \cup S_{2,1})\}$. Similarly $\lambda(\eta_2 A_\infty) = 1/2\cdot\Sigma\, \varepsilon_i$ where d_i is in $\{(S_{1,1} \cup S_{1,2}) \cap (S_{2,1} \cup S_{2,2})\}$, with the signs of $S_{2,i}$, for $i = 1$ or 2, switched. Then define $(S_{h,i};S_{j,k}) = 1/2\cdot\Sigma\, \varepsilon_g$ such that one of the d_g is in $S_{h,i}$ and the other is in $S_{j,k}$. Notice then that

$$(2.1.13) \qquad 1/2 \sum_{i=1}^{2} \sum_{j=1}^{2} (S_{i,1};S_{j,2}) = \lambda - 1/2\cdot(\varepsilon_1 + \varepsilon_2).$$

The two 2-component link diagrams $\eta_2 A_0$ and $\eta_2 A_\infty$ will have linking numbers as follows:

$$2\cdot\lambda(\eta_2 A_0) = (S_{1,1};S_{1,2}) + (S_{1,1};S_{2,1}) + (S_{2,2};S_{1,2}) + (S_{2,2};S_{2,1})$$

and

$$2\cdot\lambda(\eta_2 A_0) = (S_{1,1};S_{2,1}) - (S_{1,1};S_{2,2}) - (S_{2,1};S_{1,2}) + (S_{2,2};S_{1,2}).$$

Hence

$$\lambda(\eta_2 A_0) - \lambda(\eta_2 A_\infty) = \lambda - 1/2\cdot(\varepsilon_1 + \varepsilon_2)$$

This completes the proof of the lemma. □

Proof of 2.1.11 (continued): Using 2.1.12 we have

$$\begin{aligned}\tau(A_0) &= -\tau(\xi_2 A_\infty) - (-\varepsilon_2)\,\lambda(\eta_2 A_\infty) + 2\cdot[\tau(B_1) + \tau(B_2)]\\ &\quad + 1/2\cdot(\lambda - \varepsilon_2)(\lambda - \varepsilon_2 - \varepsilon_1) + \varepsilon_2[\lambda - 1/2\cdot(\varepsilon_1 + \varepsilon_2)]\\ &= -\tau(A_\infty) + 2\cdot[\tau(B_1) + \tau(B_2)] + 1/2\cdot\lambda\cdot(\lambda - \varepsilon 1).\end{aligned}$$ □

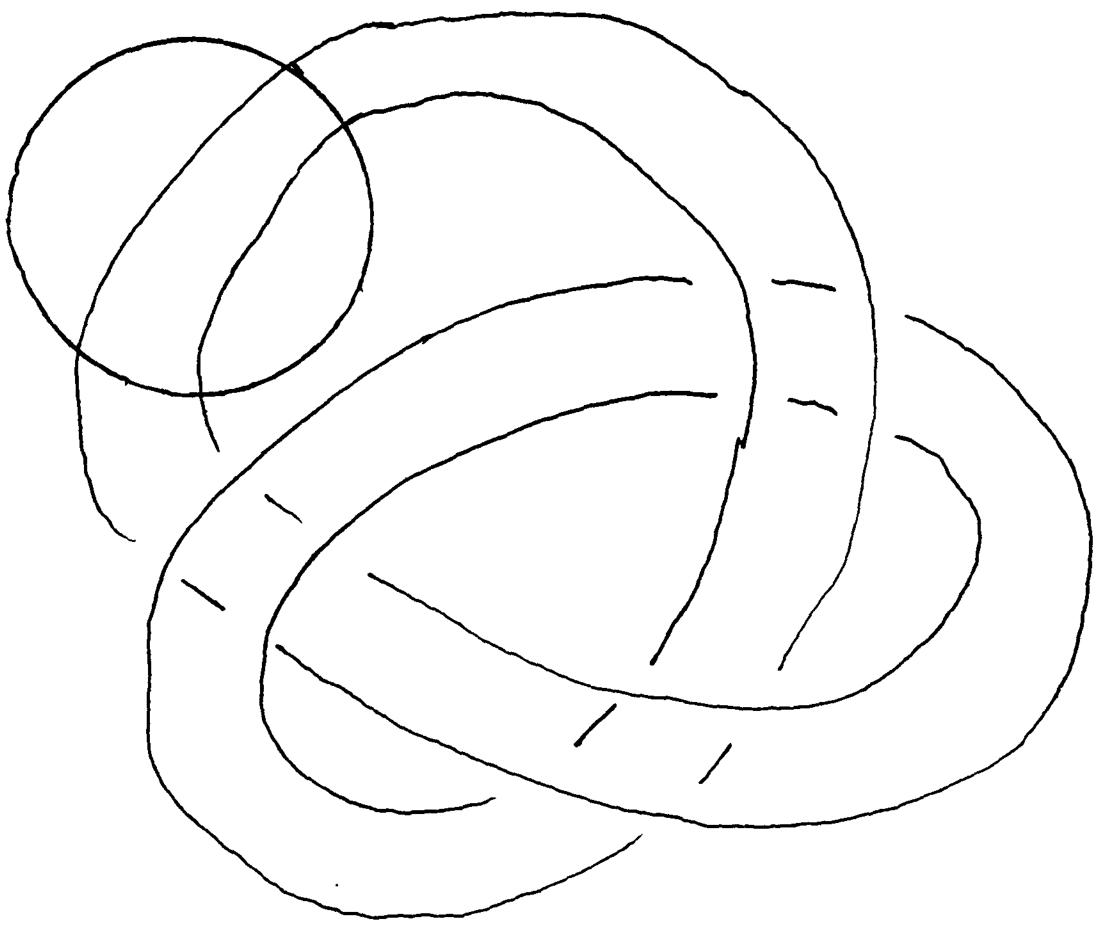
2.1.14 Corollary: Let $A_0{}^q$ be a diagram of the $(2,2q-1)$ cable knot based on its companion represented by A_0, q positive (see [**Rol**]). Then

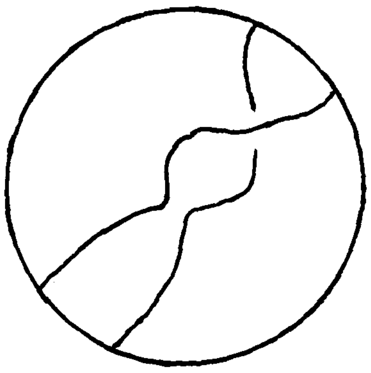
$$\tau(A_0{}^q) = 1/2\cdot q\cdot(q-1) + 4\cdot\tau(A_0).$$

Proof: Let A^q be the diagram representing the $(2,2q)$ cable link, so that $\lambda(A) = q$. Choose d to be a crossing corresponding to one of the $2q$ half twists of A. Then ηA represents $A_0{}^q$ and $\eta\tilde{A}$ represents the trivial knot. Notice that ηA^q and $\eta\tilde{A}^q$ satisfy the conditions of proposition 2.1.11. The corollary is then immediate upon its application. □

The (2,6) cable link of the trefoil.

A

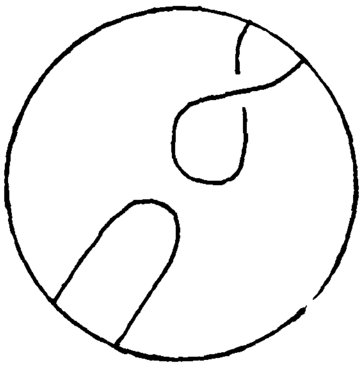




ηA

The (2,7) cable knot

of the trefoil



$\eta\tilde{A}$

The unknot

2.2 Total Twisting of a Link

In this section we consider $a_2(A)$ of $\tilde{\nabla}_A(z)$ $(= \nabla_A(z)/z^{c-1})$ where A is a c-component link diagram. Just as for knots, $a_2(A)$ may also be calculated using the following

2.2.1 Proposition: Given a c-component link diagram, c>1, and a sequence of crossing switches $\xi_1, \xi_2, \ldots, \xi_s$ from sign ε_i to $-\varepsilon_i$ which unlink one component of A (up to isotopy) there is an associated sequence of (c-1)-component link diagrams, A_j of the form

$$A_j = (\eta_j \prod_{i<j} \xi_i)A \ .$$

We then have that

$$a_2(A) \text{ of } \tilde{\nabla}_A(z) = -\sum_j \varepsilon_j \cdot a_2(A_j) \ .$$

Proof: By 1.4.2, $\tilde{\nabla}_A(z) = -\sum_j \varepsilon_j \cdot \tilde{\nabla}_A(\dot{z})$ and the result follows. □

To avoid confusion we make the following definition.

2.2.2 Definition: Let A be a c-component link diagram. Then define $T_d(A)$ as

(i) $\sum_{j=1}^{s} \varepsilon_j \cdot T_{d-1}(A_j)$ if $c = d > 1$, where A_j is as in 2.2.1

(ii) $\tau(A)$ as defined in [**LM1**] if $c = d = 1$,

(iii) 0 if $c \neq d$.

Hence, if A is a c-component link diagram and $a_2(A)$ is the coefficient of z^2 in $\tilde{\nabla}_A(z)$ then

$$(2.2.3) \qquad a_2(A) = (-1)^{c-1} \cdot \tau_c(A).$$

For the remainder of this section we concentrate on $\tau_2(L)$ where L is a 2-component link.

<u>Calculation of $\tau_2(L)$</u>:

(1) Choose a diagram, A, of the link with components B_1 and B_2.

(2) Find a set of link crossings $D = \{d_1, d_2, \ldots, d_r\}$ that when switched unlink the components of A. Label the remaining crossings $d_{r+1}, \ldots, d_n$. Let crossing d_i have sign ε_i.

(3) Choose base point p_i for the component B_i and form the $\text{Sublist}(B_i)$ as defined in the last section.

(4) For each pair of crossings $d_i, d_j \in D$ consider all products $\varepsilon_i \cdot \varepsilon_j \cdot \varepsilon_k$ where

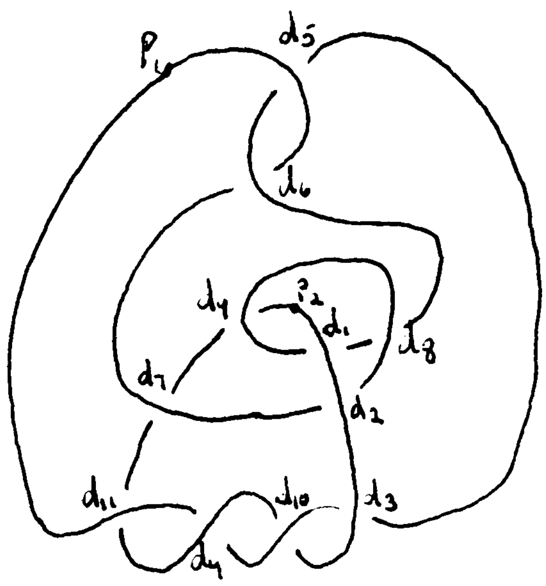
(i) d_k is a knot crossing of component B_h that appears exactly once between d_i and d_j in $\text{Sublist}(B_h)$. Denote the sum of all such products by R.

(ii) $d_k \notin D$ is a link crossing of A such that d_i, d_j, d_k appear in the same order in both sublists. [Take d_j, d_k, d_i and d_k, d_i, d_j to be the same order as d_i, d_j, d_k]. Denote the sum of all such products by S.

(iii) $d_k \in D$ such that d_i, d_j, d_k appear in the same order in both sublists. Denote this sum by T.

(6) Then $\tau_2(L) = 1/2\ (R + S - T) + \lambda(A) \cdot [\tau(B_1) + \tau(B_2)]$.

<u>2.2.4 Example</u>: Let L be represented by the diagram A below. Then



$\text{Sublist}(B_1) = d_5, d_6, d_7, d_2, d_8, d_9, d_1, d_8, d_6, d_5, d_3, d_{10}, d_4, d_{11}$,

$\text{Sublist}(B_2) = d_1, d_2, d_3, d_{10}, d_4, d_{11}, d_7, d_9$,

$R = \varepsilon_1\ \varepsilon_2(\varepsilon_8) + \varepsilon_1\ \varepsilon_3(\varepsilon_5 + \varepsilon_6 + \varepsilon_8)$
$+ \varepsilon_1\ \varepsilon_4(\varepsilon_5 + \varepsilon_6 + \varepsilon_8)$
$+ \varepsilon_2\ \varepsilon_3(\varepsilon_5 + \varepsilon_6) + \varepsilon_2\ \varepsilon_4(\varepsilon_5 + \varepsilon_6)$
$+ \varepsilon_3\ \varepsilon_4(0)$

$S = \varepsilon_1\ \varepsilon_2(\varepsilon_9) + \varepsilon_1\ \varepsilon_3(\varepsilon_7 + \varepsilon_9 + \varepsilon_{10} + \varepsilon_{11}) + \varepsilon_1\ \varepsilon_4(\varepsilon_7 + \varepsilon_9$
$+ \varepsilon_{10} + \varepsilon_{11}) + \varepsilon_2\ \varepsilon_3(\varepsilon_7 + \varepsilon_{10} + \varepsilon_{11}) + \varepsilon_2\ \varepsilon_4(\varepsilon_7 + \varepsilon_{10} + \varepsilon_{11})$
$+ \varepsilon_3\ \varepsilon_4(\varepsilon_7 + \varepsilon_9 + \varepsilon_{10} + \varepsilon_{11})$

$T = \varepsilon_1\ \varepsilon_3\ \varepsilon_4 + \varepsilon_2\ \varepsilon_3\ \varepsilon_4$ with $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_{10} = \varepsilon_{11} = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_7 = \varepsilon_8 = \varepsilon_9 = -1$. Therefore $R + S - T = -8$. Also $B_1 = B_2 = U$ so that $\tau(B_1) = \tau(B_2) = 0$. Hence, $\tau_2(A) = -4$. Notice that τ_2 detects the linking of A even though $\lambda(A) = 0$

To see how this calculation yields τ_2 as defined, consider the knot diagram $A_r = \eta_r \prod_{k=1}^{r-1} \xi_k A = \eta_r(\prod_{k=1}^{r} \xi_k)A$ where $\xi_1, \xi_2, \ldots, \xi_r$ is a sequence that unlinks the components of A. Hence A_r is simply the connected sum of knots B_1 and B_2. Similarly, if we let $A' = (\prod_{k=1}^{r} \xi_k A)$ we have that, for all $i = 1,2,\ldots,r$, $\eta_i A'$ is the connected sum of B_1 and B_2. Let A" be the ascending diagram obtained from A so that $\eta_i A''$ is a diagram of the trivial knot with the same shadow as A_i. Thus we have

$$\tau(A_i) = \tau(A_i : \eta_i A'') = \tau(A_i : \eta_i A') + \tau(\eta_i A' : \eta_i A'')$$
$$= \tau(A_i : \eta_i A') + \tau(B_1 \# B_2).$$

Noting that, $\sum_i \varepsilon_i = \lambda(A)$, we have that

$$\tau_2(A) = \sum_i \varepsilon_i \cdot \tau(A_i : \eta_i A') + \lambda(A)[\tau(B_1) + \tau(B_2)].$$

We need only show

$$\sum_i \varepsilon_i \cdot \tau(Ai : \eta_i A') = 1/2 \ (R + S - T).$$

For each i, define $D_i = \{d_{i+1}, d_{i+2}, \ldots, d_r\}$ in D. Therefore, by the definition of $\tau(A_i : \eta_i A')$, we need only show that for each $d_j \in D_i$ and $d_k \notin D_i$, the term $\varepsilon_j \ \varepsilon_k$ is a summand in either R, S or T.

For each $d_i \in D$, divide $\mathrm{Sublist}(B_1)$ into three parts, C_i, d_i and F_i, such that $C_i, d_i, F_i = \mathrm{Sublist}(B_1)$. Similarly, we have G_i and H_i with $G_i, d_i, H_i = \mathrm{Sublist}(B_2)$. Using the fact that $\mathrm{List}(A_i) = F_i, C_i, H_i, G_i$ (up to a cyclic permutation depending on the choice of base point in A_i), one can check that (4)(i) to (4)(iii) define all such $\varepsilon_j \cdot \varepsilon_k$. Notice that for $d_k \in D$, but $d_k \notin D_i$ the sign of d_k in $A_i = \eta_i \prod_{j=1}^{i-1} \xi_j A$ is $-\varepsilon_k$.

Consider now two link diagrams, A and $\tilde{A}$, where $\tilde{A}$ differs from A by a change in orientation on some of the components. The Jones polynomial, $V_A(t)$, has the property that

$$V_A(t) = t^{3d} \ V_{\tilde{A}}(t)$$

where $d = 1/2 \ [\lambda(A) - \lambda(\tilde{A})]$. This is known in the literature as the reversal theorem (see [LM1],[Mo2] and [Cm]). $\nabla_A(z)$ does not have a directly analogous property. The question arises: what is the relationship between $\nabla_A(z)$ and $\nabla_{\tilde{A}}(z)$? Below, we look at a reversing result for $\tau_2(A)$, where A is a 2-component link diagram.

2.2.5 Proposition: Let $A = B_1 \cup B_2$ be an oriented 2-component link diagram with $\lambda(A) = \lambda$. Let $\tilde{A}$ be the diagram A with the orientation on B_2 reversed, then

$$\tau_2(A) = \tau_2(\tilde{A}) + 2\cdot\lambda\cdot[\tau(B_1) + \tau(B_2)] + 1/6\cdot(\lambda^3 - \lambda).$$

Proof: The proof is by induction on the length of a sequence of crossing switches, ξ_i, $i = 1,2,\ldots,r$,that unlink the components of A. Note that when $r = 0$, $\lambda = 0$ and $\tau_2(A) = \tau_2(\tilde{A}) = 0$. Assume that switching a crossing d of sign ε yields the diagram ξA that satisfies the proposition. Let $\xi\tilde{A}$ denote the switching of the same crossing in $\tilde{A}$ and $\eta\tilde{A}$, its nullification. Then by the definition of T_2

$$\tau_2(A) = \tau_2(\xi A) + \varepsilon\ \tau(\eta A)$$

$$= \tau_2(\xi\tilde{A}) + 2\cdot(\lambda - \varepsilon)\cdot[\tau(B_1) + \tau(B_2)] + 1/6\cdot[(\lambda - \varepsilon)^3 - (\lambda - \varepsilon)] + \varepsilon\cdot\tau(\eta A).$$

Using proposition 2.1.11 we have

$$\tau(\eta A) = -\ \tau(\eta\tilde{A}) + 2\ [\tau(B_1) + \tau(B_2)] + \frac{\lambda^2 - \varepsilon\cdot\lambda}{2}.$$

Thus

$$\tau_2(A) = \tau_2(\xi\tilde{A}) + (-\varepsilon)\cdot\tau(\eta\tilde{A}) + 2\cdot\lambda\cdot[\tau(B_1) + \tau(B_2)]$$
$$+ \frac{(\lambda - \varepsilon)^3 - (\lambda - \varepsilon)}{6} + \frac{\varepsilon\cdot\lambda^2 - \lambda}{2}$$

which gives the desired result. □

<u>2.2.6 Corollary</u>: Let K^q be the $(2,2q)$ cable link of K (see corollary 2.1.14), with q a positive integer, then

$$\tau_2(K^q) = \tau_2(K^0) + 4\cdot q\cdot\tau(K) + \left[\begin{matrix} q+1 \\ 3 \end{matrix}\right]$$

<u>Proof</u>: We need only note that $\tau_2(\tilde{K}^q)$ depends only on K. That is, if $\tilde{K}^q$ denotes K^q with one of its components with reversed orientation then $\tau_2(\tilde{K}^q)$ is constant for all values of q since nullifying any of the $2q$ half twists in a diagram representing $\tilde{K}^q$ yields the trivial knot. Therefore $\tau_2(\tilde{K}^q) = \tau_2(\tilde{K}^0)$. The corollary is then obtained from two applications of the proposition since $\lambda(K^q) = q$. □

<u>2.2.7 Corollary</u>: If L and $\tilde{L}$ are two 2-component links that differ only by the orientation on one component then $\tau_2(L) - \tau_2(\tilde{L}) \equiv 1$ (modulo 2) if and only if $\lambda(L) \equiv 2$ (modulo 4).

<u>Proof</u>: By the proposition we need only state that $1/6\cdot(\lambda^3 - \lambda)$ is odd only when $\lambda \equiv 2$ (modulo 4) where $\lambda = \lambda(L)$. □

<u>2.2.8 Proposition</u>: Let A and B be c- and d-component link diagrams, respectively. Denote by $\bar{A}$ the obverse of A and let $\Omega_{c-1}(A) = (-1)^{c-1}\cdot a_0(A)$ and $\Omega_{d-1}(B) = (-1)^{d-1}\cdot a_0(B)$ where $a_0(A)$ and $a_0(B)$ are the constant terms in the normalized Conway polynomials of A and B. Then

(i) $a_2(A) = \tau_c(\bar{A})$

(ii) $\tau_c(A \# B) = \Omega_{c-1}(A)\cdot\tau_d(B) + \Omega_{d-1}(B)\cdot\tau_c(A)$

<u>Proof of (i)</u>: Use induction on $c = c(A)$. Let $\xi_1, \xi_2, \ldots, \xi_r$ be a sequence of crossings switches that unlinks one component of $A = B_1 \cup B_2 \cup \ldots \cup B_c$. For $c = 1$, (i) is true by proposition 2.1.2 and corollary 2.1.8. By the definition of $\tau_c(A)$ we have that

$$\begin{aligned}\tau_c(A) &= \sum_j \varepsilon_j\cdot\tau_{c-1}(A_j)\\ &= (-1)^{c-1}\sum_j -\varepsilon_j\cdot\tau_{c-1}(\bar{A}_j)\\ &= (-1)^{c-1}\cdot\tau_c(\bar{A}).\end{aligned}$$

Hence, $a_2(A) = (-1)^{c-1}\cdot\tau_c(A) = (-1)^{2(c-1)}\cdot\tau_c(\bar{A}) = \tau_c(\bar{A})$.

<u>Proof of (ii)</u>: Adapting P6 we have

$$\tilde{\nabla}_{A\#B}(z) = \tilde{\nabla}_A(z)\cdot\tilde{\nabla}_B(z)$$

so that

$$a_2(A\#B) = a_0(A)\cdot a_2(B) + a_0(B)\cdot a_2(A).$$

Since $c(A\#B) = c + d - 1$ and for any link diagram D $a_2(D) = (-1)^{c(D)-1}\cdot\tau_{c(D)}(D)$ we arrive at the desired result. □

<u>Remark</u>: In [Ho] Hoste uses linking numbers to calculate Ω_{c-1}. Our next section is devoted to a generalization of this result.

In general for link diagrams A, B , $A \cong B$ does not imply that for any knot diagram C, $A \# C \cong B \# C$, however, because the polynomials of these diagrams will always be the same, so too will τ_c.

2.2.8 (ii) implies that for an achiral link L with an even number of components, $\tau_c(L) = 0$. If K is a knot and K^0 is as in 2.2.6 then K achiral implies K^0 achiral. Therefore, K is chiral if $\tau_2(K^0)$ is not 0. The two-variable polynomial detects chirality and one may ask if there exists a knot K such that $P(K)(\ell,m)$ is symmetric in ℓ and $-\ell^{-1}$ but $\tau_2(K^0)$ is not 0? I guess not and conjecture that $\tau_2(K^0) = 0$ for all knots K. If this is the case, then by 2.2.6 $\tau_2(K^q)$ depends only on $\tau(K)$ and q. A stronger version of the question above asks whether there exists K such that $P(K)(\ell,m)$ is symmetric in ℓ and $-\ell^{-1}$ but $P(K^0)(\ell,m)$ is not?

2.3 $Q_{g,h}(\ell)$

One of the main problems in knot theory at present is to interpret the coefficients of the new polynomials in terms of the geometry of the link. In this section we attempt to contribute to this study.

For a link diagram A write $P(A)(\ell,m) = \sum_{j=1-c}^{M} P_j(A)(\ell)\, m^j$ where $c = c(A)$, M is the maximum power of m in $P(A)(\ell,m)$ and $P_j(A)(\ell)$ is a polynomial in ℓ and ℓ^{-1}. Recall that $P_j(\ell) = 0$ for $j \equiv c$ (modulo 2) (by P2.), for $j < 1 - c$ (also by P2.) and for $j > n - [s(A) - 1]$, where n is the number of crossings in A and s(A) is the number of Seifert circles (by P7.).

Since the substitution, $\ell = 1$, $m = z$, gives the Conway polynomial we have that

$$P_k(A)(1) = a_k(A)$$

in $\nabla_A(z)$. Hence, $P_k(A)(1)$ yields no new information. However, by factorizing out the obvious zeros from the polynomials $P_k(A)(\ell)$ for $-c < k < c-2$, $k \equiv c-1$ (modulo 2) we do obtain nontrivial invariants.

Start by rewriting the polynomial, $P(L)(\ell,m)$, of a link L in a more natural form. Recall, $P(L)(\ell,m)$ is defined recursively using the fundamental relation. At each step of the calculation, the polynomial of a link diagram is determined by the polynomials of two "less complex" link diagrams, as described in section 1.2. The result is an expression involving only ascending diagrams of varying numbers of components, each with a coefficient in $\mathbb{Z}[\ell^{\pm 1},m]$.

To each ascending diagram with c components, the polynomial μ^{c-1}, $\mu \in \mathbb{Z}[\ell^{\pm 1},m^{-1}]$, is assigned. It is here that we halt the process and look at coefficients of $\mu^g\, m^h$. Formally, we have

<u>2.3.1 Definition</u>: Let A be a link diagram with crossings d_j of sign ϵ_j for $j = 1,2,\ldots,n$. Given a tree of sequences, let $T(A) = \{D_1, D_2, \ldots, D_q\}$ be the set of all ascending diagrams of the form $D_i = \eta_{1_i}\cdots\eta_{r_i}\xi_{(r+1)_i}\cdots\xi_{s_i}A$ such that if $c_i = c(D_i)$ then

$$P(A)(\ell,m) = \sum_{i=1}^{q} [r_i]\cdot\ell^{t_i}\cdot m^{r_i}\cdot\mu^{c_i-1}$$

for some $t_i \in \mathbb{Z}$ and $[r_i] = \pm 1$, where $\mu = -m^{-1}(\ell - \ell^{-1})$.

Define

$$Q_{g,h}(A)(\ell) = \sum [r_i]\cdot\ell^{t_i}$$

taken over all $i = 1,2,\ldots,q$ such that $c_i - 1 = g$ and $r_i = h$.

2.3.2 Notes: (i) $c_i \geqslant c(A) - r_i$ for all i, hence $Q_{g,h}(A)(\ell) = 0$ for $g + h \leqslant c(A)$.

(ii) In general, $Q_{g,h}(\ell)$ depends on the tree of sequences chosen, however, as we shall demonstrate, for certain values of g and h, $Q_{g,h}(1)$ is an invariant of knot type.

Key to this section is the following.

(2.3.3) $$\sum_j (-\ell + \ell^{-1})^j \; Q_{j,j+k}(A)(\ell) = P_k(A)(\ell)$$

(1) for $k \geqslant c - 1$:

$$\sum_{j\geqslant 0} (-\ell + \ell^{-1})^j \; Q_{j,j+k}(A)(\ell) = Q_{o,k}(A)(\ell) + \sum_{j>0} (-\ell + \ell^{-1})^j \; Q_{j,j+k}(A)(\ell).$$

This yields

(2.3.4) $$P_k(A)(1) = Q_{o,k}(A)(1)$$

Hence $Q_{o,h}(A)(1)$ for $h \geqslant c-1$ are merely the coefficients of the Conway polynomial.

(2) For $1 - c \leqslant k \leqslant c - 1$ and $k \equiv c - 1 (\text{modulo } 2)$, let $a = 1/2\,(c - 1 - k)$. Then by 2.3.2 (i)

$$\sum_{j\geqslant a} (-\ell + \ell^{-1})^j \; Q_{j,j+k}(A)(\ell) = (-\ell + \ell^{-1})^a \; Q_{a,a+k}(A)(\ell)$$

$$+(-\ell + \ell^{-1})^a \cdot \sum_{j>a} (-\ell + \ell^{-1})^{j-a} \cdot Q_{j,j+k}(A)(\ell).$$

Therefore,

(2.3.5) $$(-\ell + \ell^{-1})^{-a} \; P_k(A)(\ell)\Big|_{\ell=1} = Q_{a,a+k}(A)(1).$$

Summarizing, we have the following

2.3.6 Proposition: For any c-component link diagram, A, with $0 \leqslant g,h \in \mathbb{Z}$ such that $g + h = c - 1$ or $g = 0$ and $h \geqslant c - 1$

(1) $Q_{g,h}(A)(1)$ is invariant of link type,

(2) $Q_{g,h}(A)(1) = Q_{g,h}(\xi A)(1) - \varepsilon \cdot Q_{g,h-1}(\eta A)(1)$. □

In particular,

2.3.7 Proposition: For any c-component link diagram

$$(-\ell + \ell^{-1})^{1-c} P_{1+c}(A)(\ell)\Big|_{\ell=1} = 1 \qquad \text{(cf[LM1])}$$

Proof: Let D_1 be the one element in $T(A)$ with the same shadow as A. Notice that $r_1 = 0$ and $c_1 = c(A)$ so that

$$Q_{c-1,o}(A)(1) = [r_1] = \prod_{j=1_1}^{r_1} -\varepsilon_j = 1$$ □

In [**Ho**] Hoste obtains formulae for the constant term in the Conway polynomial of a link L using both the linking matrix of L and a graphical representation of the link. It is the latter which we investigate now.

Let L be a link with components $K_1, K_2, \ldots, K_c$. Let $G(L)$ be the complete graph with c vertices. Label the vertices $K_1, K_2, \ldots, K_c$ and label the edge connecting K_i to K_j with their linking number, $\lambda(K_i, K_j)$. Now let $H_b(L)$, $0 \leqslant b \leqslant c-1$ be the set of all subgraphs of $G(L)$ that (1) consist of b distinct edges together with their vertices and (2) contain no loops. (Note that we allow elements of H_b to be disconnected. For the case $b = c-1$

though, $g \in H_{c-1}$ if and only if g is connected). If $g \in H_b$, let $\bar{g}$ be the product of the b linking numbers associated to the edges of g. Define $\Omega_b(L) = \sum_{g \in H_b} \bar{g}$ and $\Omega_0(L) = 1$.

Hoste shows that if L is an oriented link with c components, then $\tilde{\nabla}_L(0) = \Omega_{c-1}(L)(-1)^{c-1}$. We now show that $\Omega_b(L)$ is encoded in $P(L)(\ell,m)$ for all $0 \leqslant b \leqslant c-1$ and hence generalizes Hoste's result.

<u>2.3.8 Theorem</u>: If L is a c-component link then

$$(-1)^{b} \cdot (-\ell + \ell^{-1})^{b-c+1} \cdot P_{2b-c+1}(L)(\ell)\Big|_{\ell=1} = \Omega_b(L).$$

<u>Proof</u>: Using 2.3.5 we need only prove the following

$$(-1)^{b} \cdot Q_{c-1-b,b}(L)(1) = \Omega_b(A).$$

The proof is by induction on the pair (c,s) ordered lexicographically, where c is the number of components and s is the length of a sequence of crossing switches, $\xi_1, \xi_2, \ldots, \xi_s$ that unlink all the components of A. By 2.3.7, $Q_{c-1,0}(L)(1) = 1$ for all c. Setting $c = 1$ starts the induction on c. Now assume A is a link diagram representing L with pair value $(c,0)$. Then $Q_{c-1-b,b}(A)(1) = Q_{c-1-b,b}(\sqcup B_i)(1)$ where $\sqcup B_i$ is the separated union of all the components of A. This implies $\lambda(B_i, B_j) = 0$ for all i,j $(i \neq j)$. Clearly any sequence of switches and nullifications on this diagram will not decrease the number of components so that $Q_{c-1-b,b}(A)(1) = 0$ for all $b > 0$ and 2.3.7 proves the case when $b = 0$.

Let $A = \cup B_i$, $i=1,2,\ldots,c$, be an oriented c-component link diagram with a sequence of s crossing switches, ξ_j, $j = 1,2,\ldots,s$, that unlink the components of A. Let ξA and ηA be diagrams that satisfy the proposition and that differ at a single crossing d of components B_1 and B_2 (say) where d has sign ε. Denote by C_i, for $i = 1,2,\ldots,c$, the components of ξA and D_i, $i = 3,4,\ldots,c$, the components of ηA that correspond to B_i in A, with D_{12} the component of ηA that results from nullifying d. The fundamental relation gives

$$Q_{c-1-b,b}(A)(1) = Q_{c-1-b,b}(\xi A)(1) - \varepsilon\cdot Q_{c-b-1,b-1}(\eta A)(1).$$

By the induction, noting that $c(\eta A) = c(A) - 1$, we have

(2.3.9) $\quad Q_{c-1-b,b}(A)(1) = \Omega_b(\xi A) + \varepsilon\cdot\Omega_{b-1}(\eta A).$

<u>2.3.10 Notes</u>:

(1) $\lambda(B_i,B_j) = \lambda(C_i,C_j)$ for all i,j except the pairs 1,2 or 2,1, where $\lambda(B_1,B_2) = \lambda(C_1,C_2) + \varepsilon$.

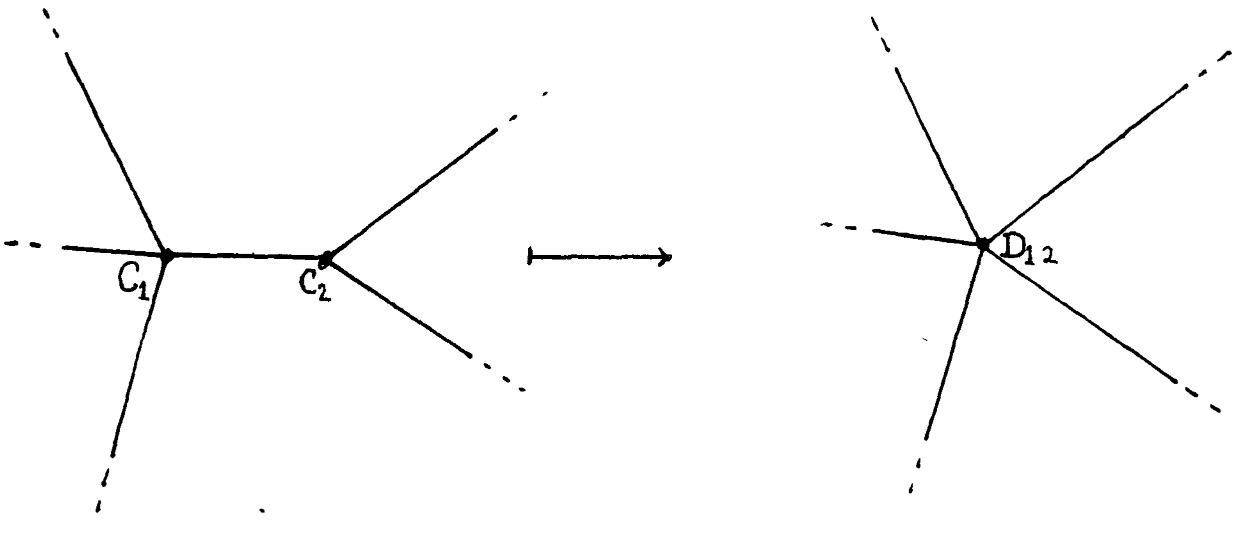
(2) $\lambda(D_{12},D_j) = \lambda(C_1,C_j) + \lambda(C_2,C_j)$ for all $j = 3,4,\ldots,c$. and $\lambda(D_i,D_j) = \lambda(C_i,C_j)$ otherwise.

By note (1) we need only look at the graphs $g \in H_b(\xi A)$ that include the edge between C_1 and C_2. (The products corresponding to those without this edge will be the same for both $\Omega_b(A)$ and $\Omega_b(\xi A)$). Denote this set of graphs by $H_b'(\xi A)$. Then

$$\sum_{g\in H_b'(\xi A)} \bar{g} = W\cdot\lambda(C_1,C_2)$$

for some W. Notice that if $\Omega_{b-1}(\eta A)$ in (2.3.9) is equal to W we use note (1) again to complete the proof. So it remains to show that $\Omega_{b-1}(\eta A) = W$.

Let g be any graph in $H_b'(\xi A)$. Construct a new graph g' using the following map: Consider $F\colon H_b'(\xi A) \longrightarrow H_{b-1}(\eta A)$ defined on the vertices by $C_1 \mapsto D_{12}$, $C_2 \mapsto D_{12}$ and $C_i \mapsto D_i$ otherwise. On the edges, send $\lambda(C_1,C_2)$ to 0 and $\lambda(C_i,C_j)$ to $\lambda(F(C_i),F(C_j))$ otherwise.



Now if $m \geqslant 0$ is the number of edges incident at D_{12} in some $g' \in H_{b-1}(\eta A)$ then 2^m graphs g in $H_b'(\xi A)$ map to g' by F. Using note (2) we have that

$$\bar{g}'\cdot\lambda(C_1,C_2) = \sum_{g\in F^{-1}(g')} \bar{g}.$$

Because F is onto we have $\Omega_{b-1}(\eta A) = W$ completing the proof. □

2.4 The Arf Invariant of an oriented Link

In [**Rob**] Robertello introduces a modulo 2 invariant of Knot cobordism (now known as the Arf invariant of a Knot K, denoted Arf(K)) and shows that it can be calculated from the Alexander polynomial. It was then shown by Kauffman that the first coefficient of the Conway polynomial is all that need be examined to determine Arf(K), [**K5**]. Conway himself along with Gordon in [**CG**] show that for Conway diagrams A_+, A_- and A_0 where A_+ and A_- represent knots

$$\mathrm{Arf}(A_+) \equiv \mathrm{Arf}(A_-) + \lambda(A_0) \pmod 2.$$

Hence, it is easily seen that

$$\tau(K) \equiv \mathrm{Arf}(K) \pmod 2$$

and the following are immediate from Proposition 2.1.11 and Corollary 2.1.14.

2.4.1 Proposition: Given the conditions of Proposition 2.1.11 and Corollary 2.1.14

(1) $\tau(A_0) + \tau(A_\infty) \equiv 1/2\cdot\lambda\cdot(\lambda - \varepsilon) \pmod 2$

(2) $\tau(A_0{}^q) \equiv 0 \pmod 2$ if and only if $q \equiv 2$ or $3 \pmod 4$ □

We turn our attention now to the Arf invariant of Links. First some necessary definitions.

2.4.2 Definition: Let L be a link with components $K_1, K_2, \ldots, K_c$. L is said to be proper if for any choice of i, $\lambda(L, L-K_i) = \sum_{j\neq i} \lambda(K_i, K_j)$ is an even integer. Now let L be represented by the diagram A. Another link L' represented by A' is said to be <u>related to L</u> if A' is obtained from A by smoothing a link crossing and links related to the same link are themselves said to be <u>related</u>.

<u>2.4.3 Proposition</u> (Robertello): If knots K and K' are both related to the same proper link L then Arf(K) = Arf(K'). □

Hence, one defines the Arf invariant of a proper link to be the Arf invariant of any knot to which it is related. Now let L' be a sublink of a link L for which both L' and L − L' are themselves proper and denote this by L' < L. Wu [Y] proves the following.

<u>2.4.4 Theorem</u> (Ying-Qing): For any c-component link L,

$$\tau_c(L) \equiv \sum_{L' < L} \mathrm{Arf}(L') \quad (\text{modulo } 2).$$

In particular, for 2-component links of the form $L = K_1 \cup K_2$,

$$\mathrm{Arf}(L) \equiv \tau_2(L) + \tau(K_1) + \tau(K_2) \quad (\text{modulo } 2).$$ □

Wu's theorem actually uses $a_2(L)$ of $\tilde{\nabla}_A(z)$ instead of $\tau_c(L)$ but by (2.2.3) the two are congruent modulo 2.

The aim now is to use Wu's Theorem to obtain results regarding the effect orientation of link components has on the Arf invariant. Our first result is the following proposition about $\tau_c(L)$.

<u>2.4.5 Proposition</u>: If $A = B_1 \cup B_2 \cup \ldots \cup B_c$ and $\tilde{A} = B_1 \cup \ldots \cup B_{c-1} \cup \tilde{B}_c$ are c-component link diagrams, where $\tilde{B}_i$ indicates the orientation on the i-th component is reversed, then $\tau_c(A) - \tau_c(\tilde{A})$ (modulo 2) is a function of the linking numbers, $\lambda(B_i, B_j)$ for all i, j, i<j.

Proof: Our induction is on c, the number of components and on r, the number of pairs i,j such that B_i is nontrivially linked with B_j in A (B_i is trivially linked with B_j if at every crossing of B_i and B_j, B_i passes over B_j for $i<j$.) Note that ηA and $\eta\tilde{A}$ will not have the same shadow if the link crossing operated on involves the component whose orientation is being changed. Consider first then the case where A has no linking other than on its c-th component, i.e. $A - B_c = \bigsqcup_{i=1}^{c-1} B_i$ (the separated union of the first c-1 components of A).

Let $D_i = \tilde{B}_1 \cup \ldots \cup \tilde{B}_{i-1} \cup B_i \cup \ldots \cup B_c$ so that $A = D_1$ and $\tau_c(\tilde{A}) = \tau_c(D_c)$ (by P3) and hence we need only show that $\tau_c(D_i) - \tau_c(D_{i+1})$ (modulo 2) is a function of the linking. The difference between D_i and D_{i+1} is the orientation on the i-th component. In this case we can easily find a sequence of switches that unlinks a component from B_c without involving the i-th component of Di, for $c>2$. By reordering the components, the hypotheses of the proposition will be satisfied and crossings operated on will not involve the component whose orientation is being changed.

Corollary 2.2.7 shows that for 2-component link diagrams $\tau_2(A) - \tau_2(\tilde{A}) \equiv 1$ (modulo 2) if and only if $\lambda(A) \equiv 2$ (modulo 4) which starts the induction on c. Also $\tau_c(\bigsqcup_{i=1}^{c} B_i) \equiv 0$ (modulo 2), by C2, which starts the induction on r.

Assume now that $c > 2$ and $r > 0$. Choose a sequence of crossing switches ξ_i, $i=1,2,\ldots,s$ that unlinks B_i from B_j for some pair i,j, $i\neq j$, and reduces the value r above. Denote the resulting diagram by D. The definition of $\tau_c(A)$ gives

$$\tau_c(A) = \tau_c(D) + \sum_{k=1}^{s} \varepsilon_k \, \tau_{c-1}(A_k).$$

Likewise

$$\tau_c(\tilde{A}) = \tau_c(\tilde{D}) + \sum_{k=1}^{s} \varepsilon_k \, \tau_{c-1}(\tilde{A}_k).$$

where $A_k = (\eta_k \prod_{h<k} \xi_h)A$ and $\tilde{A}_k = (\eta_k \prod_{h<k} \xi_h)\tilde{A}$. The induction gives

that for each k, $\tau_{c-1}(A_k) - \tau_{c-1}(\tilde{A}_k)$ (modulo 2) is a function of their linking. However the linking numbers of A_k are constant for all $i = 1,2,\ldots,s$. Combining we have

$$\tau_c(A) - \tau_c(\tilde{A}) = \tau_c(D) - \tau_c(\tilde{D}) + \lambda(B_i, B_j)\cdot[\tau_{c-1}(A_1) - \tau_{c-1}(\tilde{A}_1)].$$

Since the nontrivial linking of both D and A_1 are determined by the linking of A the induction holds and the proof is complete. □

Using Wu's theorem we have the following corollary.

2.4.6 Corollary: The change in the Arf invariant of a proper link due to a change in the orientation on some of its components is a function of the linking numbers.

Proof: Wu notes that theorem 2.4.4 can be used inductively to calculate the Arf invariant of a link from $\tau_c(L')$ where $L' < L$ and hence, by 2.4.5 a change in the Arf invariant due to a change in orientaion is a funtion of the linking numbers of all such L'. However, any linking number of such an L' is a linking number of L.□

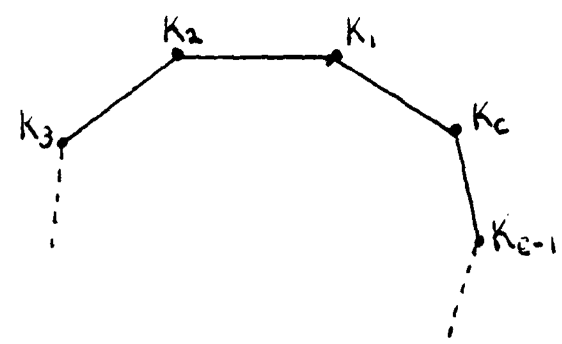
Let L be a c-component link and $G(L)$ its graph (as in section 2.3). Let $M(L)$ be the graph obtained from $G(L)$ by reducing each $\lambda(B_i, B_j)$ modulo 2 and removing all edges whose linking is trivial. Murakami, in [**Muk**], shows that if $L = K_1 \cup K_2 \cup \ldots \cup K_c$ is such that $M(L)$ is a tree with no even edges, then $\tau_c(L)$ depends only on the (mod 2) total twisting of each K_i. Noting that the total twisting of a knot does not depend on the orientation we see that $\tau_c(L)$ for such a link is independent of the orientations on its components. However such links are never proper so that analogous results for the Arf invariant are not possible. A category of links that is proper and whose Arf invariant is independent of the orientation is the purely proper links. A link $L = K1 \cup K2 \cup \ldots \cup Kc$ is said to be <u>purely proper</u> if $\lambda(Ki, Kj)$ is even for all i, j, $i \neq j$.

<u>2.4.7 Proposition</u>: If L is a purely proper c-component link with $c \geqslant 3$ then $Arf(L)$ does not depend on the orientation of its components.

<u>Proof</u>: Since all sublinks of a purely proper link are themselves purely proper, we need only show that $\tau_c(L) \equiv \tau_c(\tilde{L})$ (modulo 2). By reconstructing the proof 2.4.5 and noting that $\lambda(K_i, K_j) \equiv 0$ (modulo 2) we see that the induction clearly goes through. □

To demonstrate a link whose Arf invariant is dependent on the orientation we turn once again to Murakami.

2.4.8 Proposition: Let $L = K_1 \cup K_2 \cup \ldots \cup K_c$ be a c-component link such that $M(L)$ is as shown below, where each $\lambda(K_i, K_j)$ is odd.



Then $\operatorname{Arf}(L) = \tau_c(L) + (n+1)\left[\sum_{i=1}^{c} \tau(K_i)\right]$ □

2.4.9 Corollary: Let $A = B_1 \cup B_2 \cup \ldots \cup B_c$, $c > 2$, represent a link L such that $M(L)$ is as shown above. Let $\tilde{A} = B_1 \cup B_2 \cup \ldots \cup B_{c-1} \cup \tilde{B}_c$ be the diagram A with the orientation on the c-th component reversed. Then $\operatorname{Arf}(A) \equiv \operatorname{Arf}(\tilde{A})$ (modulo 2) if and only if $\lambda(B_{c-1}, B_c) \equiv \lambda(B_c, B_1)$ (modulo 4), i.e. $\lambda(A, A-B_c) \equiv 2$ (modulo 4).

Proof: Using the above proposition, once again we need only prove the result for $\tau_c(A)$. Induct on the number of components. Choose a sequence of crossing changes $\xi_1, \xi_2, \ldots, \xi_s$ that unlinks B_i from B_{i+1}, $i \neq c$ or $c-1$. Denote this diagram by D. Then

$$(2.4.10) \qquad \tau_c(A) = \tau_c(D) + \sum_{k=1}^{s} \varepsilon_k \cdot \tau_{c-1}(A_k).$$

Notice that $M(D)$ is a tree and by the above comments $\tau_c(D) \equiv \tau_c(\tilde{D})$ (modulo 2) for all $c \geqslant 3$. Also, for $c>3$, A_k satisfies the hypotheses of the proposition for all $k = 1,2,\ldots,s$, so that by the induction

$$\tau_{c-1}(A_k) \equiv \tau_{c-1}(\tilde{A}_k) \text{ (modulo 2)}$$

if and only if $\lambda(A, A-B_c) \equiv 2$ (modulo 4). Placing this information into (2.4.10) we have

$$\tau_c(A) \equiv \tau_c(\tilde{A}) \text{ (modulo 2)}$$

if and only if $\lambda(B_i, B_j)$ is odd and $\lambda(A, A-B_c) \equiv 2$ (modulo 4)
But, because $\lambda(B_i, B_{i+1})$ was chosen odd the induction holds.

For the case $c = 3$, A_k is a 2-component link diagram and by Corollary 2.2.7, the initial condition is satisfied and the proof is complete. □

Finally we construct a formula for determining if a link is proper or not.

<u>2.4.11 Theorem</u>: Let L be a c-component link. Then L is a proper link if and only if

$$\sum_{b=1}^{c-1} \Omega_b(L) = 0 \text{ (modulo 2)}.$$

<u>Proof</u>: Let L be represented by the link diagram $A = B_1 \cup B_2 \cup \ldots \cup B_c$. If A is a purely proper link, that is, $\lambda(B_i, B_j)$ is even for all i, j, $i \neq j$, then by the definition of both proper links and $\Omega_b(A)$ the result follows. Induct, then, on the number of $\lambda(B_i, B_j)$ that are odd.

Given Conway diagrams A_+, A_- and A_0 where A_+ and A_- differ at a link crossing we have that either none or exactly two of the three diagrams is proper. This is true since exactly one of A_+ or

A_- is proper if and only if A_0 is proper and neither is proper otherwise. The information in section 2.4 gives

$$\Omega_b(A_+) \equiv \Omega_b(A_-) + \Omega_{b-1}(A_0) \pmod{2}$$

Operating on a link crossing of A between two components whose linking is odd and noting that for any link diagram D, $\Omega_0(D) = 1$ and $\Omega_c(D) = 0$, we have that

$$\sum_{b=1}^{c-1} \Omega_b(A_+) = \sum_{b=1}^{c-1} \Omega_b(A_-) + \sum_{b=1}^{c(A_0)-1} \Omega_b(A_0) + 1 \pmod{2}$$

and the result follows. □

CHAPTER 3

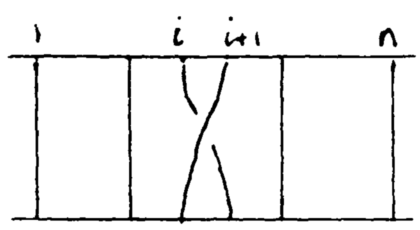
3.1 The braid group, B_n

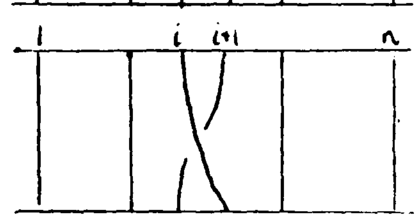
The classical n-string braid group, B_n [Ar], is the group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ subject to the relations

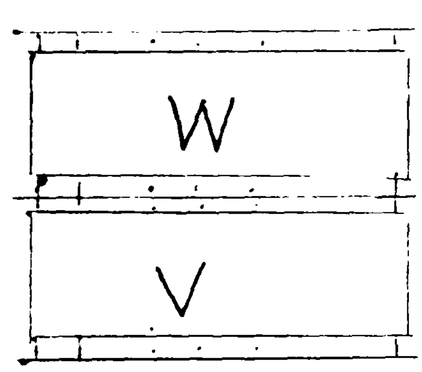
B1. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $1 \leqslant i \leqslant n-2$

B2. $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| \geqslant 2$.

The elements (<u>n-braids</u>) of the braid group B_n are thus equivalence classes of words (<u>braid words</u>) in the generators and their inverses. Each braid word corresponds to a unique oriented braid diagram as follows:

σ_i ←------→ 

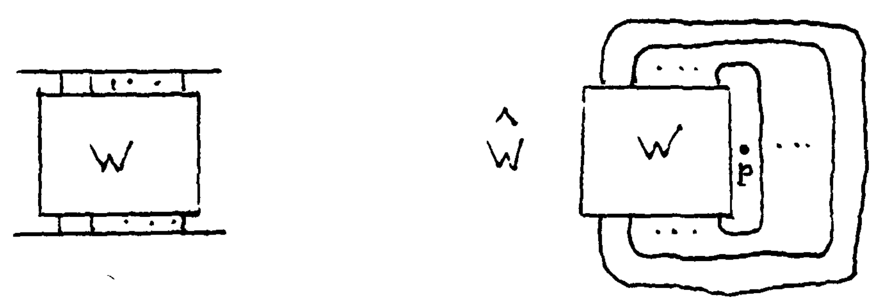
σ_i^{-1} ←------→ 

W·V ←------→ 

In the braid diagram corresponding to the braid word $W = \prod_{i=1}^{k} \sigma_{\theta_i}^{\varepsilon_i}$, where $1 \leqslant \theta_i \leqslant n-1$, $\varepsilon_i = \pm 1$ for each i, we call the crossing corresponding to $\sigma_{\theta_i}^{\varepsilon_i}$ the <u>i-th crossing</u> and say that it is in the <u>θ_i-th column</u>.

For two braids $\beta, \gamma \in B_n$ we write $\beta \equiv \gamma$ if β, γ are conjugate in B_n. For two braid words W, V we write $W = V$ if W, V are identical, $W \simeq V$ if W, V represent the same braid and $W \equiv V$ if W, V represent braids that are conjugate in B_n.

To every braid diagram W one can form a closed braid diagram $\hat{W}$ (or $(W)^{\wedge}$) by connecting n arcs so that there exists a point p called the axis of $\hat{W}$ with the property that $\hat{W}$ rotates around p in a positive direction as in the figure below.



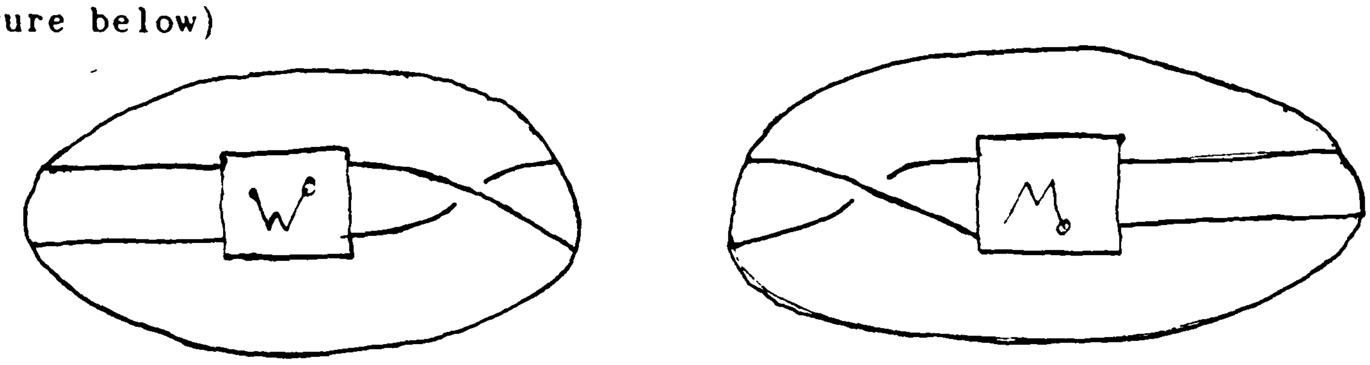
A closed braid represents a link, indeed an oriented link, since the braid strings are oriented from top to base. Markov [**Ma**] in 1935 showed that every link can be represented by a closed braid diagram and that closed braid diagrams $\hat{W}, \hat{V}$ give the same link if and only if W can be obtained from V by a sequence of the following equivalences (now called Markov moves):

M1. $W \equiv V$;

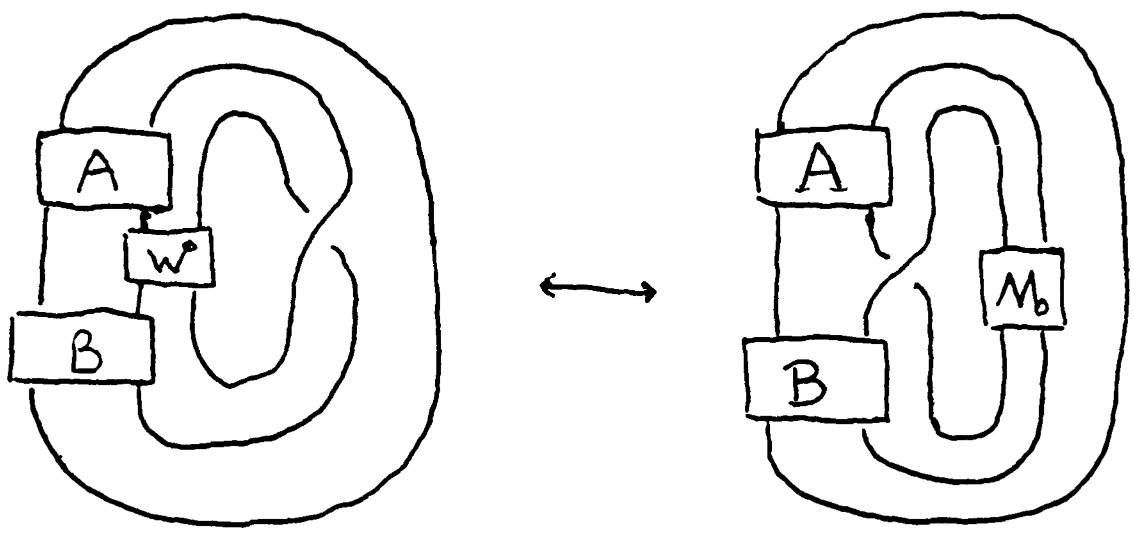
M2. $W \simeq V\,\sigma_{n-1}^{\pm 1}$, where W and V represent braids in B_n and B_{n-1} respectively.

Details of the theory of braids quoted may be found in [**Bi**].

One standard means of obtaining braids that belong to a different conjugacy class but whose closed braids give the same link is by flyping. A <u>flype</u> is a move on a tangle (with two inputs and two outputs) obtained by rotating the tangle 180° [K3,p.27]. (See figure below)

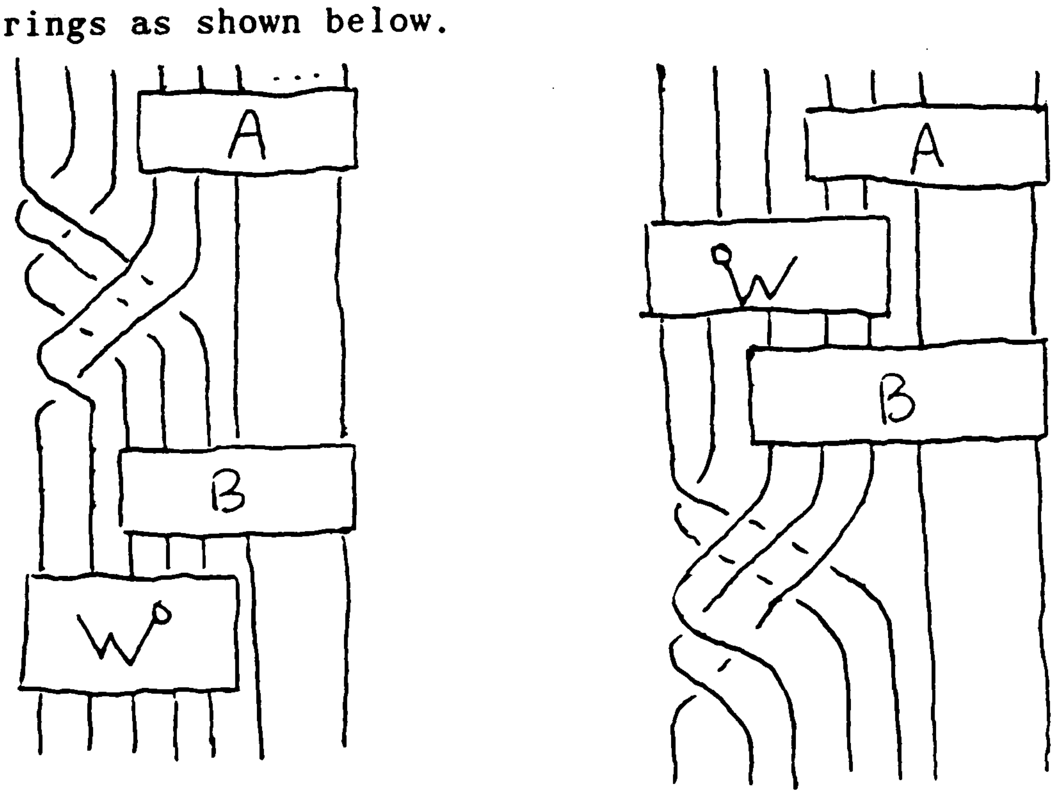


By flyping one has that if A and B are two braid diagrams on only the first n-1 strings then the closures of $A\sigma_{n-1}^{k}B\sigma_{n-1}^{\varepsilon}$ for $k \in \mathbb{Z}$, $\varepsilon = \pm 1$ and $A\sigma_{n-1}^{\varepsilon}B\sigma_{n-1}^{k}$ represent the same link [Mus]. (See figure)



Later in this chapter we shall prove by example using a new invariant that these two braids need not be conjugate.

Notice that this extends to flypes of tangles of more than two strings as shown below.



3.2 The Φ invariant of braid conjugacy class

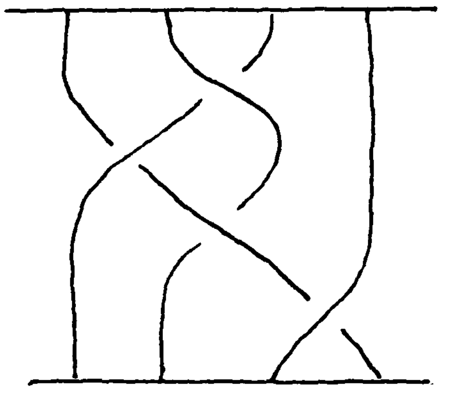
To define this invariant we represent each braid word W in a new way, its lifting form, as follows. Label the strings in the braid diagram $1, 2, \ldots, n$ in order according to their starting points. The braid word W determines a permutation, e_W say, of $\{1, 2, \ldots, n\}$, where $e_W(i)$ is the end-point of the i-th string. We then (uniquely) represent W by the expression

$$\prod_{i=1}^{k} g_{\theta_i, \varphi_i}^{\varepsilon_i}$$

if at the i-th crossing, string θ_i crosses string φ_i, for $\theta_i < \varphi_i$, where $\varepsilon_i = \pm 1$ is the sign of the i-th crossing in W.

Example 1. $W = \sigma_i^{\varepsilon}$ has lifting form $g_{i,i+1}^{\varepsilon}$, $\varepsilon = \pm 1$.

Example 2. $W = \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_3 \in B_4$ has lifting form $g_{2,3}^{-1} g_{1,3} g_{1,2}^{-1} g_{1,4}$. (See figure below).

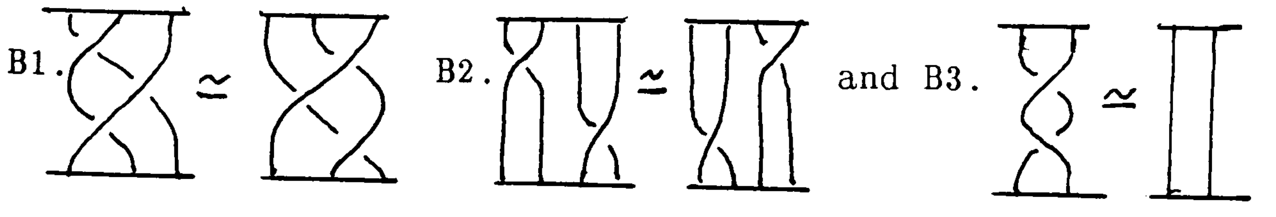
Braid word W	Braid diagram	Lifting form
$\sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_3 \in B_4$		$g_{2,3}^{-1} g_{1,3} g_{1,2}^{-1} g_{1,4}$

Next we abelianize the product $\prod_{i=1}^{k} g_{\theta_i,\varphi_i}^{\varepsilon_i}$ to obtain the form

(3.2.1) $$\Gamma_1\,\Gamma_2\ldots\Gamma_{n-1}$$

where $\Gamma_i = \prod_{j=i+1}^{n} g_{i,j}^{\delta_{i,j}}$ for $1 \leqslant i < j \leqslant n$. Notice that under the braid relations

B1. B2. and B3.



this "abelianized" lifting form remains unchanged and so is an invariant of the braid class. In the case when the braid is a pure braid, i.e. e_W is the identity permutation, the abelianized lifting form is also invariant under conjugation up to a reordering of the $g_{i,j}$, although this is not true in general.

To overcome this drawback we next, in 3.2.1, identify $g_{i,j}$ with $g_{k,\ell}$ if $e_W(i), e_W(j)$ are either k,ℓ or ℓ,k. The expression in 3.2.1 then results in an expression $\Phi(W)$ in the $g_{i,j}$. Suppose V is any braid word. Then $\Phi(V^{-1}WV)$ can be obtained from $\Phi(W)$ by replacing $g_{i,j}$ by $g_{e_V(i),e_V(j)}$ for each i,j. Thus up to reordering of the generators $g_{i,j}$, Φ determines an invariant of the braid conjugacy class.

<u>Example 2</u>. (continued from above) The braid word W has abelianized lifting form

$$g_{1,2}^{-1}\,g_{1,3}^{+1}\,g_{1,4}^{+1}\,g_{2,3}^{-1}\,g_{2,4}^{0}\,g_{3,4}^{0}\,.$$

Under the identification $g_{1,2} \sim g_{2,4} \sim g_{2,3} \sim u$ (say) and $g_{1,3} \sim g_{1,4} \sim g_{3,4} \sim v$ (say) then

$$\Phi(W) = u^{-2}v^{2}.$$

Example 3. In [G], Garside defines the fundamental word of B_n, Δ_n, as follows: let Π_s be the word consisting of the generators $\sigma_1, \sigma_2, \ldots, \sigma_s$ listed in ascending order so that

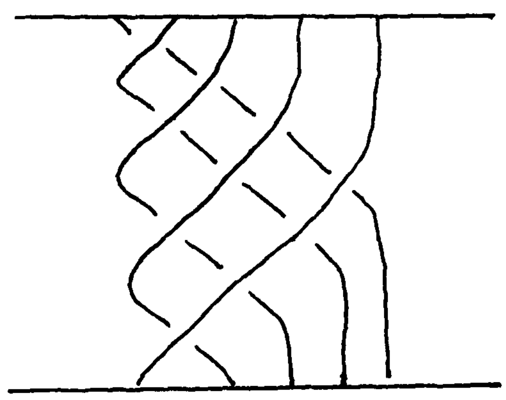
$$\Pi_s = \prod_{i=1}^{s} \sigma_i .$$

Then define

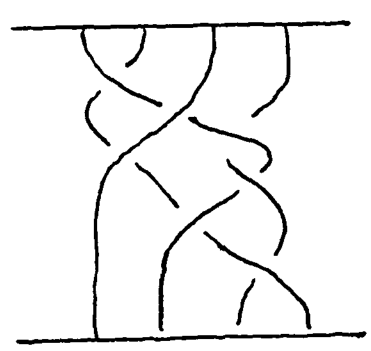
$$\Delta_n \equiv \Pi_{n-1} \Pi_{n-2} \ldots \Pi_1 .$$

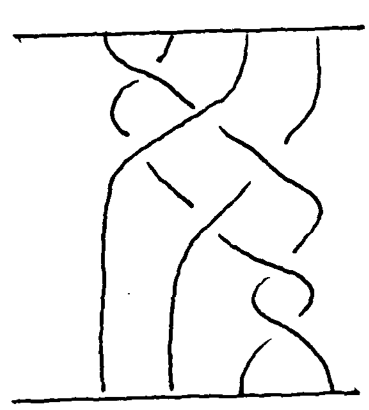
The abelianized lifting form of $\Delta_n \in B_n$ written as in 3.2.1 has $\delta_{i,j} = 1$ for all $1 \leqslant i < j \leqslant n$.

Δ_5



Example 4. Each of the following braid diagrams yields the trefoil knot. (The diagrams differ by a flype of the crossings in column 3.) However, Φ shows that the two are not conjugate.

W =  and $\Phi(W) = (u_1^0 v_1^{-1})$

V =  and $\Phi(V) = (u_2^{-3} v_2^2)$

Example 5. If W is a braid word on n strings such that $\hat{W}$ is a knot diagram then $\Phi(W)$ takes the form $u_1^{\alpha_1} u_2^{\alpha_2} \dots u_k^{\alpha_k}$ $(\alpha_i \in \mathbb{Z})$ where

$$k = \begin{cases} (n-1)/2 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

In particular if $n = 3$, $\Phi(W) = u^{\omega(\hat{W})}$.

3.3 Ascending braid diagrams

3.3.1 Definition. A braid diagram W is ascending with base points $k_1, \dots, k_c$ if $\hat{W}$ is an ascending link diagram with base points at the top of the k_i-th strings, $i = 1, \dots, c$ (where c is the number of components in $\hat{W}$).

3.3.2 Proposition. Let W be a braid diagram with n strings such that $\hat{W}$ represents a knot. If W is ascending with base point k, the writhe of $\hat{W}$ is given by

$$\omega(\hat{W}) = n + 1 - 2k.$$

Proof. The hypothesis that W is an ascending braid diagram with base point k means that the k-th string always passes under all other strings and the $e_W^{-1}(k)$-th string always passes over all other strings. In general the $e_W^{i}(k)$-th string will always pass over the $e_W^{j}(k)$-th string for $0 \leqslant j < i < n$. For this reason we may think of each string as being in its own "plane" in which it may "move" independently of all other strings.

It follows that if braid diagram V also satisfies the hypothesis of the proposition and $e_V = e_W$ then $V \simeq W$. Noting that the exponent sum is not affected by the braid relation, this implies $\omega(\hat{V}) = \omega(\hat{W})$.

To establish the proposition we now use induction on the number of strings n. For $n = 1,2$ the proposition is easily verified. For the inductive step we suppose W has n strings and the proposition has been established for smaller values of n. For ease of notation set $a = e_W^{-1}(n)$, $b = e_W(n)$ and note that $b \neq n$ since $\hat{W}$ is assumed to represent a knot.

Let U be any braid diagram on $n-1$ strings satisfying the following axioms:

1. $e_U(i) = \begin{cases} e_W(i) & \text{for } i = 1,2,\ldots,a-1,a+1,\ldots,n; \\ b & \text{for } i = a \end{cases}$
2. U is an ascending braid diagram with base point k, if $k < n$, and base point b, if $k = n$.

Note. It is possible to construct such a diagram by nullifying a crossing of the a-th and n-th strings in W (one must exist since $e_W(a) = n$) and then removing the n-th string entirely. Also, if two such diagrams exist they must be equal in B_n by the above statements.

Let U' be the diagram U with an extra trivial string attached so that $e_U'(n) = n$.

Define a braid V on n strings as follows.

If $k = n$, take

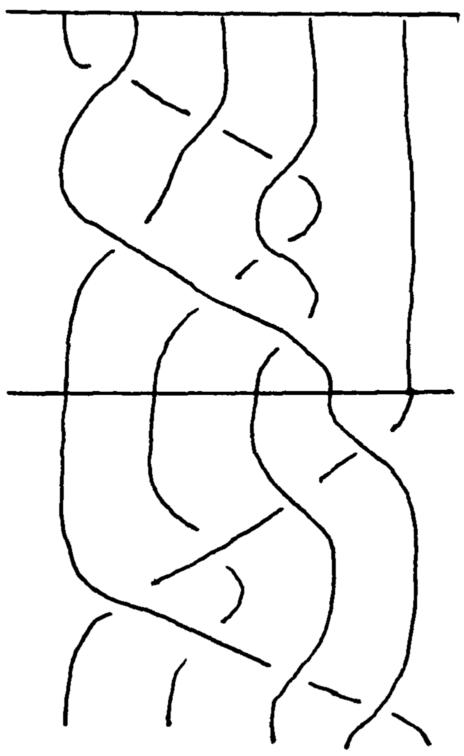
$$V = \sigma_{n-1}^{-1}\,\sigma_{n-2}^{-1}\cdots\sigma_{b+1}^{-1}\,\sigma_b^{-1}\,\sigma_{b+1}^{-1}\cdots\sigma_{n-2}^{-1}\,\sigma_{n-1}^{-1}\ ;$$

if $k < n$, take

$$V = \sigma_{n-1}^{\varepsilon_{n-1}}\,\sigma_{n-2}^{\varepsilon_{n-2}}\cdots\sigma_{b+1}^{\varepsilon_{b+1}}\,\sigma_b\,\sigma_{b+1}^{-\varepsilon_{b+1}}\cdots\sigma_{n-2}^{-\varepsilon_{n-2}}\,\sigma_{n-1}^{-\varepsilon_{n-1}},$$

where

$$\varepsilon_i = \begin{cases} +1 & \text{if } i \text{ preceeds } b \text{ in the ordered list} \\ & \{ e_U(k), e_U^2(k), \ldots, e_U^{n-1}(k) = k \} \\ -1 & \text{otherwise.} \end{cases}$$



Obverse of U'·V

We note that U,U' and V are such that

1. U satisfies the induction;

2. $e_V(b) = n$, $e_V(n) = b$ and $e_V(i) = i$ for $i \neq b,n$ so that $e_{UV}(i) = e_V \circ e_U\ (i) = e_W(i)$;

3. $U' \cdot V$ is an ascending braid diagram with a base point k.

Hence $U' \cdot V \simeq W$. But

$$\omega(U' \cdot V) = \omega(U') + \omega(V) = \begin{cases} (n-2k) + (1) & \text{for } n > k \\ (n-2b) + (-2(n-1-b)-1) & \text{for } n = k. \end{cases}$$

Therefore $\omega(W) = n+1-2k$. □

<u>3.3.3 Corollary</u>. Let W be any braid diagram, then

$$\omega(\hat{W}) \equiv n - c(\hat{W}) \pmod 2$$

where $c(\hat{W})$ is the number of components in $\hat{W}$.

Proof: Since changing the sign of any crossing leaves $\omega(\hat{W})$ fixed modulo 2, we have that $\omega(\hat{W}) \equiv \omega(\hat{V})$ where V is an ascending diagram obtained from W.

Let $n(c_i)$ be the number of strings that make up the i-th component of $\hat{V}$ and $sw(c_i)$ be the sum of signs of all crossings between the i-th component of $\hat{V}$ and itself, then

$$\sum_i n(c_i) = n$$

and

$$\sum_i sw(c_i) = sw(\hat{V}).$$

Since V is assumed to be ascending we have, by 1.1.1, $sw(\hat{V}) = \omega(\hat{V})$, since $\lambda(\hat{V}) = 0$. By use of the proposition on the i-th component considered separately as a braid whose closure is a knot we have

$$sw(c_i) \equiv n(c_i) + 1 \text{ (modulo 2)}.$$

Then summing over all components

$$\omega(\hat{V}) \equiv n + c(\hat{V}) \text{ (modulo 2)}.$$

Hence $\omega(\hat{W}) \equiv n + c(\hat{W})$ (modulo 2), since $c(\hat{W}) = c(\hat{V})$ and no term in this expression is affected by the braid relations. □

3.3.4 Corollary. Let W be any braid diagram. If $|\omega(\hat{W})| \leqslant n - c(\hat{W})$ then one can find crossings $d_1,\dots,d_{2q}$ of W such that

(1) the crossings alternate in sign;

(2) when the signs of all the crossings are changed, the resulting braid diagram is ascending for some base points $b_1,\dots,b_k$, where $k = c(\hat{W})$.

Proof: We first prove the corollary for the case when $c(\hat{W}) = k = 1$, i.e. $\hat{W}$ is a knot diagram. Consider the ascending diagram $\hat{V}$ with the same knot shadow as $\hat{W}$ and base point at the top of the $\frac{1}{2}[n + 1 - \omega(\hat{W})]$-th string. Notice that such a string exists if and only if $|\omega(\hat{W})| \leq n - 1$, since $\omega(\hat{W}) \equiv n + 1$ (modulo 2) by corollary 3.3.3. By the proposition $\omega(\hat{V}) = \omega(\hat{W})$ and so the number of positive and negative crossings that must be changed in converting W to V must be equal. By their independence we may order them so that they alternate in sign.

Suppose now $c(\hat{W}) = k > 1$. Consider again V such that $\hat{V}$ is an ascending diagram with the same link shadow as $\hat{W}$. For reasons analogous to the knot case we need only show $\omega(\hat{V}) = \omega(\hat{W})$. Choose a_i in $\mathbb{Z}$, for $i = 1,2,\dots,c(\hat{W})$ such that:

(1) $|a_i| \leq n(c_i) - 1$

(2) $a_i \equiv n(c_i) + 1$ (modulo 2)

(3) $\Sigma\, a_i = \omega(\hat{W})$

where $n(c_i)$ is the number of strings that make up the i-th component of $\hat{V}$.

Note: Such a_i must exist by corollary 3.3.3 since $\omega(\hat{W}) = \Sigma\, a_i \equiv \Sigma\, n(c_i) + 1 = n + c(\hat{W})$ (modulo 2).

Consider c_i, the i-th component of V, as a separate ascending braid diagram. If c_i has base point $b_i = \frac{1}{2}[n(c_i) + 1 - a_i]$ then $\omega(\hat{c}_i) = a_i$ by the proposition. Therefore,

$$\omega(\hat{W}) = \Sigma\, a_i = \Sigma\, \omega(\hat{c}_i) = sw(\hat{V}) = \omega(\hat{V})$$

since $\lambda(\hat{V}) = 0$. □

In [**Mol**] a braid $\gamma \in B_n$ is termed <u>reducible</u> if $\gamma \equiv \beta\sigma_{n-1}^{\pm 1}$ for some $\beta \in B_{n-1}$. By use of the Markov moves, if a link has a reducible representative in B_n then it also has a representative in B_{n-1}. The following corollary shows that by Markov moves that do not increase string number, any ascending braid W has an equivalent braid in $B_{c(\hat{W})}$ (cf. [**Mol**]).

<u>3.3.5 Corollary</u>. Let W be an ascending braid diagram such that the closed braid $\hat{W}$ represents a knot. Then the braid represented by W is reducible. Moreover, there exist braid diagrams S,T on $n-1$ strings such that $W \simeq S\,\sigma_{n-1}^{\pm 1}\,T$, where either $S\cdot T$ or $T\cdot S$ is equivalent to an ascending braid diagram.

<u>Proof</u>. We observe that for U',V in the proof of the proposition the braid relations yield

$$V \simeq T_1^{-1}\,\sigma_{n-1}\,T_1 = (\sigma_b^{-\varepsilon_{b+1}}\sigma_{b+1}^{-\varepsilon_{b+2}}\cdots\sigma_{n-2}^{-\varepsilon_{b-1}})\sigma_{n-1}(\sigma_{n-2}^{+\varepsilon_{b-1}}\cdots\sigma_b^{+\varepsilon_{b+1}})$$

for $n > k$ and

$$V \simeq \bar{T}_2^{-1}\,\sigma_{n-1}\,T_2 = (\sigma_b^{-1}\,\sigma_{b+1}^{-1}\cdots\sigma_{n-2}^{-1})\,\sigma_{n-1}^{-1}\,(\sigma_{n-2}^{-1}\cdots\sigma_b^{-1})$$

for $n = k$, where $\bar{T}_2$ is the obverse of T_2. Thus

$$W \simeq U'\cdot V \simeq U'\cdot T_1^{-1}\,\sigma_{n-1}\,T_1 \qquad ,\text{for } n > k$$

and

$$W \simeq U'\cdot V \simeq U'\cdot \bar{T}_2^{-1}\,\sigma_{n-1}\,T_2 \qquad ,\text{for } n = k.$$

and these have a single crossing in the $n-1$ column and are therefore reducible.

Let $S_1 = U'\cdot T_1^{-1}$ and $S_2 = U'\cdot \bar{T}_2^{-1}$ then;

(1) for $n \geqslant k$, $S_1\cdot T_1 \simeq U'$ and U' is by construction ascending,

(2) for $n = k$ note first that because $U'\cdot V \simeq U'\cdot\bar{T}_2\,\sigma_{n-1}^{-1}\,T_2$ are ascending with base point n then $\sigma_{n-1}^{-1}\,T_2\cdot U'\cdot\bar{T}_2^{-1}$ is ascending

with base point n. Hence $T_2 \cdot S_2$ is ascending with base point $n - 1$. □

Note: By induction, any ascending braid diagram in B_n, whose closure is a knot, has an equivalent braid (by Markov moves) in B_1. If its closure is a link with c components, then the braid is equivalent to a braid in B_c.

CHAPTER 4

4.1 Chirality and the polynomials

This section focuses on chirality and the ability the Jones polynomial, $V(t)$, and its two variable generalization, $P(\ell,m)$ have in detecting it.

By their construction $P(K)(\ell,m)$ and $V(t)$ are identical to $P(\bar{K})(-\ell^{-1},m)$ and $V_{\bar{K}}(-t^{-1})$, so that if $P(K)(\ell,m)$ is not symmetric in ℓ and $-\ell^{-1}$ or $V_K(t)$ is not symmetric in t and $-t^{-1}$, then the knot K is chiral (see P4. in section 1.3).

Unfortunately the converse is not true. The following two results show that if a link can be represented by a closed 3-braid diagram $\hat{W}$ such that $\omega(\hat{W}) = 0$ then both $P(\hat{W})(\ell,m)$ and $V_{\hat{W}}(t)$ are symmetric in ℓ and $-\ell^{-1}$ and t and $-t^{-1}$, repectively. Many chiral knots and links hold this property. The knot 10_{48} in the Alexander-Briggs notation is the first of these. Note, though, that this knot's chirality is detected by the Kauffman polynomial (see [K4]).

4.1.1 Proposition: If W is any braid diagram on three strings such that $\omega(\hat{W}) = 0$ then $P(\hat{W})(\ell,m)$ is symmetric in ℓ and $-\ell^{-1}$.

Proof: We prove 4.1.1 by induction on the pair (r,s), ordered lexicographically, where $2r$ is the number of crossings in a diagram and $2s$ is, using corollary 3.3.4, the number of crossing switches in a sequence ξ_i which result in an ascending diagram with writhe zero (for some choice of base point and ordering of the components). The statement is true for $r = 0,1$ and 2, as all the corresponding links are achiral. It is also trivially true for the pair $(r,0)$, for all $r \geq 0$.

Notation: If W is the braid diagram

$$W = \prod_{i=1}^{2n} \sigma_{\theta_i}^{\varepsilon_i} \qquad , \ 1 \leq \theta_i \leq n-1,\ \varepsilon_i = \pm 1$$

then let $(\xi_i^{\varepsilon_i} W)$ be the diagram W with the i-th crossing switched from sign ε_i to sign $-\varepsilon_i$ and let $(\eta_i^{\varepsilon_i} W)$ be the diagram W with the i-th crossing nullified. Notice that the writhes of these diagrams are $\omega(\hat{W}) - 2\varepsilon_i$ and $\omega(\hat{W}) - \varepsilon_i$, respectively. For the remainder of this section, we take advantage of the correspondence between braid diagrams and closed braid diagrams and write $P(W)(\ell,m)$ for the HOMFLY polynomial of $\hat{W}$.

Now let $W = \prod_{i=1}^{2r} \sigma_{\theta_i}^{\varepsilon_i}$ be a braid diagram such that $\Sigma\, \varepsilon_i = \omega(\hat{W}) = 0$ and $\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_s}$ is a sequence of switches that yield an ascending diagram of zero writhe. Recall by corollary 3.3.4 we can choose these crossings to alternate in sign. Bearing this in mind, apply the fundamental relation to a positive crossing of W;

$$P(W) = \ell^{-2}\, P(\xi_{i_1}^{+1}\, W) - \ell^{-1}\, m\, P(\eta_{i_1}^{+1}\, W).$$

Then operating on a negative crossing in the sequence yields

$$\ell^{-2}\, P(\xi_{i_1} W) = P(\xi_{i_2}^{-1} \xi_{i_1} W) + \ell^{-1}\, m\, P(\eta_{i_2}^{-1} \xi_{i_1} W).$$

Choosing the corresponding crossing of $(\eta_{i_1}\ W)$ and applying the relation gives

$$-\ \ell^{-1}\, m\, P(\eta_{i_1} W) = -\ \ell m\, P(\xi_{i_2}^{-1} \eta_{i_1} W) - m^2\, P(\eta_{i_2}^{-1} \eta_{i_1} W),$$

so that

$$P(W) = P(\xi_{i_2}^{-1}\xi_{i_1}W) + m\,[\ell^{-1}\,P(\eta_{i_2}^{-1}\xi_{i_1}W) - \ell\,P(\xi_{i_2}^{-1}\eta_{i_1}W)$$
$$- m^2\,P(\eta_{i_2}^{-1}\eta_{i_1}\,W)$$

By the induction $P(\xi_{i_2}^{-1}\xi_{i_1}W)$ and $m^2\,P(\eta_{i_2}^{-1}\eta_{i_1}W)$ are symmetric in ℓ and $-\ell^{-1}$. Therefore $P(W)$ is symmetric if and only if

(4.1.2) $$\ell^{-1}\,P(\eta_{i_2}^{-1}\xi_{i_1}W) - \ell\,P(\xi_{i_2}^{-1}\eta_{i_1}W)$$

is symmetric.

Using corollary 3.3.4 on $\eta_{i_2}^{-1}\xi_{i_1}W$ and $\xi_{i_2}^{-1}\eta_{i_1}W$ we can find sequences of $2q$ crossings for each such that when their signs are changed they yield ascending diagrams with writhes of -1 and $+1$,respectively. By corollary 3.3.3, these diagrams must have 2 components each.

Remark: In general the sequence lengths $2q_1$ and $2q_2$,as defined in 3.3.4, will not be equal, i.e. $q_1 < q_2$ (say); but by changing the sign of a single crossing $2(q_2 - q_1)$ times we can define $q = q_2$ without affecting the proof at all.

The effort we have put into ensuring that our sequences of crossing switches alternate in sign allows us to define the following sequences of diagrams;

$$U_k = (\xi_{j_k}^{+\varepsilon}\,\xi_{j_{k-1}}^{-\varepsilon}\cdots\xi_{j_2}\,\xi_{j_1}^{-1}\,\eta_{i_2}^{-1}\xi_{i_1}\,W)$$
$$V_k = (\xi_{h_k}^{-\varepsilon}\,\xi_{h_{k-1}}^{+\varepsilon}\cdots\xi_{h_2}^{-1}\xi_{h_1}\,\xi_{i_2}^{-1}\eta_{i_1}\,W)$$

for $k = 0, i,\ldots,2q$, and where $\varepsilon = +1$ or -1 if k is even or odd, respectively. Notice that $\omega(U_k) = -\,\omega(V_k) = -\varepsilon$. Hence

$$(4.1.3) \qquad \ell^{-\varepsilon} P(U_k) = \ell^{\varepsilon} P(U_{k+1}) + \varepsilon \cdot m\, P(\eta_{j_{k+1}}^{-\varepsilon} U_k)$$

$$(4.1.4) \qquad -\ell P(V_k) = -\ell^{-\varepsilon} P(V_{k+1}) + \varepsilon \cdot m\, P(\eta_{h_{k+1}}^{\varepsilon} V_k)$$

By the induction both $P(\eta_{j_{k+1}}^{-\varepsilon} U_k)$ and $P(\eta_{h_{k+1}}^{\varepsilon} V_k)$ are symmetric in ℓ and $-\ell^{-1}$. Hence adding (4.1.3) and (4.1.4) we have that

$$(4.1.5) \qquad \ell^{-\varepsilon} P(U_k) - \ell^{\varepsilon} P(V_k)$$

is symmetric in ℓ and $-\ell^{-1}$ if and only if

$$(4.1.6) \qquad \ell^{\varepsilon} P(U_{k+1}) - \ell^{-\varepsilon} P(V_{k+1})$$

is symmetric in ℓ and $-\ell^{-1}$. The substitution of $k = 0$ into (4.1.5) gives (4.1.2) and $k = 2q$ into (4.1.5) gives the following;

$$\ell^{-1} \mu - \ell \mu = -m^{-1} (-\ell^{-2} + 2 - \ell^{2}),$$

where $\mu = -m^{-1}(\ell - \ell^{-1})$. Thus $P(W)$ is symmetric in ℓ and $-\ell^{-1}$.

□

<u>4.1.7 Corollary</u>: If W is any braid diagram on 3 strings such that $\omega(\hat{W}) = 0$ then $V_{\hat{W}}(t)$ is symmetric in t and $-t^{-1}$.

<u>Proof</u>: The relation between $P(\hat{W})(\ell,m)$ and $V_{\hat{W}}(t)$ is as follows [L]:

$$V_{\hat{W}}(t) = P(\hat{W})(t, t^{1/2} - t^{-1/2}).$$

The corollary is then immediate from proposition 4.1.1. □

4.2 Visible achirality

Recall from section 1.1 that if A is a link diagram and $\bar{A}$ its obverse then both $A \# \bar{A}$ and $A \sqcup \bar{A}$ are achiral. We call these links <u>trivially achiral</u>. This section looks at a nontrivial construction of achiral knots and links via the braid groups. We first introduce three useful self-maps of B_n.

(1) Define Garside's [G] <u>reflection of B_n</u>, $R : B_n \longrightarrow B_n$ to be defined on the generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ by $R(\sigma_i) = \sigma_{n-i}$ and extend to yield an automorphism of B_n, and hence for braid words V, W, if $V \simeq W$ then $R(V) \simeq R(W)$ and we write $R(\beta)$ for the braid they represent.

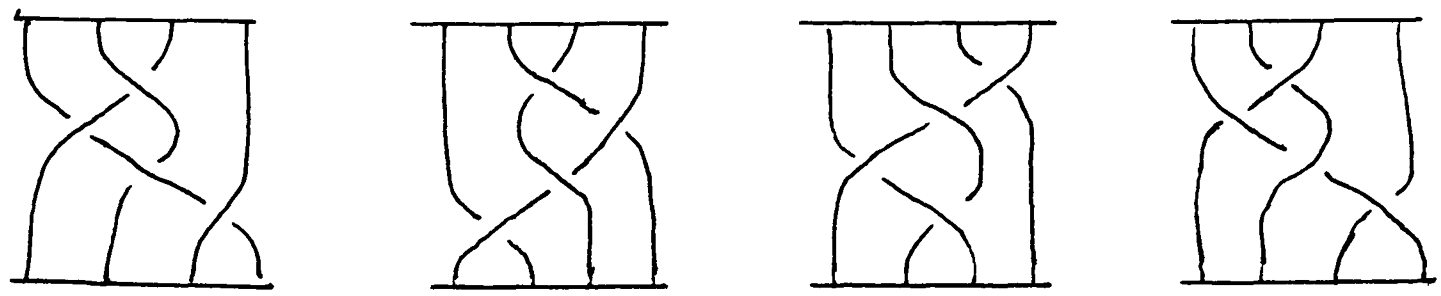
(2) Define the <u>flip of B_n</u>, $F : B_n \longrightarrow B_n$, $F(\sigma_i) = \sigma_i$ for all $i = 1, 2, \ldots, n-1$ and $F(V \cdot W) = F(W) \cdot F(V)$. This anti-isomorphism again is well defined for all $\beta \in B_n$.

(3) Consistent with previous notation, define the map

<u>Bar</u> : $B_n \longrightarrow B_n$ on the generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ by $\mathrm{Bar}(\sigma_i) = \sigma_i^{-1}$. Let $\mathrm{Bar}(W) = \bar{W}$ so that if $W = \prod_{i=1}^{k} \sigma_{\theta_i}^{\varepsilon_i}$ then $\bar{W} = \prod_{i=1}^{k} \sigma_{\theta_i}^{-\varepsilon_i}$. As with link diagrams, call $\bar{W}$ the obverse of W.

Clearly Bar is a well defined automorphism of B_n.

W $\qquad$ $R(W)$ $\qquad$ $F(W)$ $\qquad$ $\bar{W}$



(4.2.1) Notes: (1) The maps R, F, and Bar commute.

(2) The compositions $R \circ R$, $F \circ F$ and Bar $\circ$ Bar are the identity map.

(3) $F(\bar{\beta}) = \beta^{-1}$ so that $\beta \cdot F(\bar{\beta}) \simeq F(\bar{\beta}) \cdot \beta \simeq 1$ for all $\beta \in B_n$, where 1 is the identity in B_n.

(4)(a) Considered as oriented link diagrams, the two closed braid diagrams $\hat{W}$ and $R(W)^\wedge$ are isotopic, i.e. $\hat{W} \cong R(W)^\wedge$, where $R(W)^\wedge$ is the closure of $R(W)$.

(b) If $\hat{V}$ is $\hat{W}$ with its orientation reversed then $\hat{V} \cong F(W)^\wedge$.

(5) If Δ_n is the fundamental word in B_n (see example 3. in section 3.1) and W is any word in B_n, then (a) $\Delta_n \simeq R(\Delta_n)$ and (b) $\Delta_n \cdot W \simeq R(W \cdot \Delta_n)$.

Using the maps R, F and Bar we have the following lemmas.

<u>4.2.2 Lemma</u>: Let β be any braid in B_n, then the closures of the following braids are achiral;

1. $\beta \cdot \bar{\beta}$
2. $\beta \cdot R(\bar{\beta})$

Proof: (1) $\beta\cdot\bar{\beta} \equiv \bar{\beta}\cdot\beta = \overline{\beta\cdot\bar{\beta}}$ in B_n and by M2. their closures are isotopic.

(2) $$[\beta\cdot R(\bar{\beta})]^{\wedge} \cong [\ R(\ \beta\cdot R(\bar{\beta}))\ ]^{\wedge} \qquad \text{by note (4) above}$$

$$\cong [\ \overline{R(\bar{\beta})\cdot\beta}\ ]^{\wedge} \qquad \text{by note (2)}$$

$$\cong [\ \overline{\beta\cdot R(\bar{\beta})}\ ]^{\wedge} \qquad \text{by M2.} \qquad \Box$$

4.2.3 Lemma: Let β be any braid in B_n , then the closures of the following braids are reversed achiral.

1. $\beta\cdot F(\bar{\beta})$
2. $\beta\cdot F \circ (\bar{\beta})$

Proof: (1) By note (3) $\beta\cdot F(\bar{\beta}) \simeq 1$ in B_n and hence has closure isotopic to U^n which is, in fact, trivially achiral.

(2) Let $\hat{\gamma}$ be $[\beta\cdot F \circ R(\bar{\beta})]^{\wedge}$ with its orientation reversed, then

$$\hat{\gamma} \cong [F \circ R(\beta\cdot F \circ R(\bar{\beta}))\ ]^{\wedge} \qquad \text{by note (4) above}$$

$$\cong [\overline{F \circ R(\bar{\beta})\cdot\beta}\ ]^{\wedge} \qquad \text{by note (2)}$$

$$\cong [\overline{\beta\cdot F \circ R(\bar{\beta})}\ ]^{\wedge} \qquad \text{by M2.} \qquad \Box$$

4.2.4 Lemma: Let $\gamma \in B_n$ be either of the two braids in 1., or 2. of lemma 4.2.2 (respectively, 4.2.3), then the closure of γ^m is achiral (respectively, reversed achiral).

Sketch: We need only note that for any braids β, γ

$$(\beta\cdot\gamma)^m = (\beta\cdot\gamma)^{m-1}\cdot(\beta\cdot\gamma) \equiv \gamma\cdot(\beta\cdot\gamma)^{m-1}\cdot\beta = (\gamma\cdot\beta)^m$$

and that $R(\beta^m) = [R(\beta)]^m$, $F(\beta^m) = [F(\beta)]^m$ and $\bar{\beta}^{\,m} = \overline{\beta^m}$.

Repeating the steps in each of the above proofs yields the desired result. □

Suppose $W\cdot\bar{W}$ is a braid word in B_n. Then $\Delta_n\cdot W\cdot\bar{W}\cdot\Delta_n^{-1} \simeq \Delta_n\cdot W\cdot\Delta_n^{-1}\cdot R(\bar{W}) \simeq \Delta_n\cdot W\cdot R(\overline{\Delta_n\cdot W})$. Hence for any braid of the form $(\beta\cdot\bar{\beta})^m \in B_n$ for any integer m there exists $\gamma \in B_n$ such that $(\beta\cdot\bar{\beta})^m \equiv (\gamma\cdot R(\bar{\gamma}))^m$.

<u>4.2.5 Definition</u>: A braid $\gamma \in B_n$ is said to be <u>visibly achiral</u> if for some $\beta \in B_n$, $m \in \mathbb{Z}$, $\gamma \equiv (\beta\cdot R(\bar{\beta}))^m$. Likewise, γ is <u>visibly reversed achiral</u> if $\gamma \equiv (\beta\cdot F \circ R(\bar{\beta}))^m$.

For the remainder of this section we wish to investigate the set of visibly achiral and visibly reversed achiral 3-braids. We begin with the following.

<u>4.2.6 Theorem</u>: Let β be a braid in B_3. Then β is conjugate to its mirror image if and only if β is visibly achiral.

<u>Proof</u>: (<=) We have that $\beta \equiv (\gamma\cdot R(\bar{\gamma}))^m \equiv (R(\bar{\gamma})\cdot\gamma)^m \equiv \overline{(\gamma\cdot R(\bar{\gamma}))^m} \equiv \bar{\beta}$ for some $\gamma \in B_n$.

(=>) The canonical presentation of B_3 is

$$p = |\sigma_1\ \sigma_2 : \sigma_1\ \sigma_2\ \sigma_1 = \sigma_2\ \sigma_1\ \sigma_2|$$

since the relation, B2. does not apply. p is equivalent to the presentation $p' = |a,b\colon a^2 = b^3|$. (Take $a = \sigma_1\ \sigma_2\ \sigma_1$ and $b = \sigma_1\ \sigma_2$). Letting $a^2 = b^3 = c$ and noting that c generates the centre of B_3 it is clear that any word in p' must be conjugate to one and only one of the following, for some integers m, n , with $m > 0$ and both ℓ, $\ell_i = 1$ or 2 , for $i = 1, 2, \ldots, m$

1. c^n
2. $c^n\ a$
3. $c^n\ b^{\ell}$
4. $c^n\ ab^{\ell_1}ab^{\ell_2}\ldots ab^{\ell_m}$, up to a cyclic permutation of $(\ell_1, \ell_2, \ldots, \ell_m)$.

(The above work is due to Murasugi in [**Mus**])

We have that the obverses of a, b and b^2 are $\bar{a} = c^{-1}a$, $\bar{b} = c^{-2}\ ab^2\ a$ and $\overline{b^2} = c^{-2}$ aba, respectively. Hence, the obverses of each of the forms above become

1'. c^{-n}

2'. $c^{-n-1}\ a$

3'. $c^{-n-1}\ b^k$,where $k = 3 - \ell$

4'. $c^{-n-2m}\ ab^{k_1}\ ab^{k_2}\ldots ab^{k_m}$,where $k_i = 3 - \ell_i$

Assume now that $\beta \in B_3$ is conjugate to its mirror image. The conjugacy class of β has a unique representative in one of the forms 1. ,...,4. , so that if $\beta \equiv \bar{\beta}$, then both β and $\bar{\beta}$ must be represented by the same word. The only possible non-trivial case is when $\beta \equiv c^n\ ab^{\ell_1}ab^{\ell_2}\ldots ab^{\ell_m} \equiv c^{-n-2m}\ ab^{k_1}\ ab^{k_2}\ldots ab^{k_m}$. We then have that $n = -n - 2m$ and that the sequence of powers $(\ell_1, \ell_2, \ldots, \ell_m)$ is a cyclic permutation of the sequence of powers $(k_1, k_2, \ldots, k_m)$. That is, $n = -m$ and, that with subscripts considered modulo m, $\ell_i = k_{x+i}$ for some x and for all i in $\mathbb{Z}_m$. Since $n = -m$ we can rewrite β, up to conjugacy, as

$$\beta \equiv c^{-1}\, ab^{\ell_1}\, c^{-1}\, ab^{\ell_2} \ldots c^{-1}\, ab^{\ell_m}$$

Observe that $c^{-1}\, ab^2$ translates to σ_2 in p and $c^{-1}\, ab$ to $\sigma_1{}^{-1} = R(\overline{\sigma_2})$ in B_3 and that $k_i = 1,2$ according as $\ell_i = 2,1$, respectively. We complete the theorem with the aid of the following lemma.

<u>4.2.7 Lemma</u>: Given the conditions above, set $d = GCD(2x,m)$. Then

(a) m is even (hence d is even)

(b) $\ell_i = \ell_{i+d}$ for all $i \in \mathbb{Z}$

(c) $\ell_i = k_{i+\frac{1}{2}d}$ for all $i \in \mathbb{Z}$.

<u>Proof of (a)</u>: The sequence $(\ell_1, \ell_2, \ldots, \ell_m)$ is a cyclic permutation of $(k_1, k_2, \ldots, k_m)$. Hence the number of $\ell_i = 1$ must be the same as the number of $k_i = 1$. Conversely, $k_i = 3 - \ell_i$ so that the number of $\ell_i = 1$ must also be the same as the number of $k_i = 2$. Therefore m is even.

(b) $\ell_i = k_{x+i} = 3 - \ell_{x+i} = 3 - k_{2x+i} = 3 - (3 - \ell_{2x+i}) = \ell_{2x+i}$, where subscrpts are interpreted modulo m. Likewise, $\ell_i = \ell_{2qx+i}$ for all $q \in \mathbb{Z}$. The result then follows.

(c) Suppose $qd = 2x$. By (b), $k_{x+i} = \ell_i = \ell_{sd+i}$ for all $i,s \in \mathbb{Z}_m$. We note that if $q = 2s$ for some $s \in \mathbb{Z}$ then we arrive at the contradiction that $\ell_i = k_i$. Hence q is not even. We have that $\ell_i = \ell_{sd+i} = k_{sd+\frac{1}{2}qd+i} = k_{\frac{1}{2}d(2s+q)+i}$, for all $i,s \in \mathbb{Z}_m$ and can choose s so that $2s + q \equiv 1$ (modulo m). □

In conclusion

$$\beta \equiv (c^{-1}\ ab^{\ell_1} \ldots c^{-1}\ ab^{\ell_e}\ c^{-1}\ ab^{k_1} \ldots c^{-1}\ ab^{k_e})^r$$

where $e = \frac{1}{2}d$ and $rd = m$. Now if $c^{-1}ab^{\ell_i} = \sigma_{\theta_i}{}^{\varepsilon_i}$ then $c^{-1}ab^{k_i} = R(\sigma_{\theta_i}{}^{-\varepsilon_i})$ for $\theta_i = 1,2$, ε_i. Translating into p we have for some $\gamma \in B_3$,

$$\beta \equiv (\gamma \cdot R(\bar{\gamma}))^r \qquad \square$$

The analogous theorem for visibly reversed achiral 3-braids is

<u>4.2.8 Theorem</u>: Let β be a braid in B_3. Then β is conjugate to the flip of its obverse if and only if β is visibly reversed achiral.

<u>Proof</u>: (<=) Obvious.

(=>) Once again the use of the presentation p' is very helpful. (Using Garside's summit word we can arrive at the same result only after a long tedious calculation). Setting $a = \sigma_1\ \sigma_2\ \sigma_1$ and $b = \sigma_1\ \sigma_2$ we have $F(a) = a$, $F(c) = c$ and $F(b^\ell) = c^{-1}\ ab^\ell$ where $\ell = 1$ or 2. The representatives of the flipped obverse of each of the forms in 1.,...,4. are then

1". c^{-n}

2". $c^{-n-1}\ a$

3". $c^{-n-1}\ b^k$,where $k = 3 - \ell$

4". $c^{-n-2m}\ ab^{k_m}\ ab^{k_{m-1}} \ldots ab^{k_1}$,up to a cyclic permutation of $(k_m, k_{m-1}, \ldots, k_1)$, where $k_i = 3 - \ell_i$.

Assume now that $\beta \equiv F(\bar{\beta})$ so that both β and $F(\bar{\beta})$ are represented by the same word. Once again the only possible nontrivial case is when

$$\beta \equiv c^n\, ab^{\ell_1}\, ab^{\ell_2} \ldots ab^{\ell_m} \equiv c^{-n-2m}\, ab^{k_m}\, ab^{k_{m-1}} \ldots ab^{k_1}$$

This implies $n = -m$ and the sequence of powers, $(\ell_1, \ell_2, \ldots, \ell_m)$ is a cyclic permutation of sequence of powers $(k_m, k_{m-1}, \ldots, k_1)$. Considering subscripts modulo m we have for some $x \in \mathbb{Z}_m$,

(a) m is even, for the same reason m was even in the proof of 4.2.7.

(b) $\ell_i = k_{x-i}$, (see note below)

(c) x is odd since $x = 2s$ yields the contradiction

$$\ell_s = k_{2s-s}$$

(d) For $z = \frac{1}{2}(x-1)$, $\ell_z = k_{z+1}$ and $\ell_{z+\frac{1}{2}m} = k_{z+1+\frac{1}{2}m}$; simply substitute $i = z$ and $i = z + \frac{1}{2}m$ into $\ell_i = k_{x-i}$

We then have for $e = z + \frac{1}{2}m$

$$\beta \equiv (C^{-1}\, ab^{\ell_z} \ldots c^{-1}\, ab^{\ell_e}\, c^{-1}\, ab^{k_e} \ldots c^{-1}\, ab^{k_z})^r.$$

As before, if $c^{-1}\, ab^{\ell_i} = \sigma_{\theta_i}^{\ \varepsilon_i}$ then $c^{-1}\, ab^{k_i} = R(\sigma_{\theta_i}^{\ -\varepsilon_i})$. then translating β into p we have, for some $\gamma \in B_3$

$$\beta \equiv \gamma \cdot F\circ R(\bar{\gamma}).$$

□

<u>Note</u>: It is possible that x, as in (b) above, is not unique. Suppose that $\ell_i = k_{y-i}$ for some $y \neq x$ and for all $i \in \mathbb{Z}_m$. Then $\ell_i = k_{x-i} = k_{y-i}$ so that $\ell_i = \ell_{x-y+i}$. Set $d = GCD(x-y, m)$ with $rd = m$. Because m is even and x,y are both odd , d must be even. Then $\ell_i = \ell_{i+d}$ for all $i \in \mathbb{Z}_m$ and write

$$\beta \equiv (c^{-1}\, ab^{\ell_1}\, c^{-1}\, ab^{\ell_2} \ldots c^{-1}\, ab^{\ell_d})^r$$

and we may proceed with the proof above replacing "m" by "d". Notice, though, that finding at least one x is sufficient to prove the theorem.

It has been conjectured (informally, at least) that the isotopy classes of minimal string braid representatives are generated by conjugacy and flypes of the type in section 3.1. Notice that if a flype is possible on a word W in B_3, i.e. $W = \sigma_1^{\alpha_1} \sigma_2^{\alpha_2} \sigma_1^{\alpha_3} \sigma_2^{\varepsilon}$ where $\alpha_i \in \mathbb{Z}$, $\varepsilon = \pm 1$ then $\hat{W} \cong [F(W)]^\wedge$ since $\hat{W}$ is also represented by $W' = \sigma_1^{\alpha_1} \sigma_2^{\varepsilon} \sigma_1^{\alpha_3} \sigma_2^{\alpha_2} \equiv \sigma_2^{\varepsilon} \sigma_1^{\alpha_3} \sigma_2^{\alpha_2} \sigma_1^{\alpha_1} \equiv F(W)$. If the conjecture is true (at least for B_3) then the conjugacy classes of the 3-string braid diagrams W and $F(W)$ generate all 3-string braid representatives of $\hat{W}$ and $\hat{W}$ with its orientation reversed, given that $\hat{W}$ is not represented by a 2-braid. Thus $\hat{\beta}$, for $\beta \in B_3$ would be unorientedly achiral if and only if β is visibly unorientedly achiral. I conjecture that every (reversed) achiral link has a visibly (reversed) achiral representative in B_n for some n.

<u>4.2.9 Corollary</u>: If β is a braid in B_3 such that $\beta \equiv \bar{\beta}$ or $\beta \equiv F(\bar{\beta})$ then the closure $\hat{\beta}$ is an alternating link.

<u>Proof</u>: As seen in the proofs of 4.2.6 and 4.2.8, if β is conjugate to its mirror image then it is conjugate to a word in the generators σ_1^{-1} and σ_2. This is sufficient to show that its closure has an alternating link diagram. □

<u>4.2.10 Corollary</u>: Let W be a 3-string braid diagram such that $\hat{W}$ is an alternating closed braid diagram. Then, for some 3-string alternating diagrams V, S, T and integer m we have that

(i) if $W \equiv \bar{W}$ then $W = (V \cdot R(\bar{V}))^m$

(ii) if $W \equiv F(\bar{W})$ then $TS = (V \cdot F \circ R(\bar{V}))^m$ where $W = ST$.

Proof: $\hat{W}$ alternating implies either W or $\bar{W}$ is a word in σ_1^{-1}, σ_2. Without loss of generality we take W to be such a word. Translating into p' we have that

$$W = c^{-1}\, ab^{\ell_1} \ldots c^{-1}\, ab^{\ell_m}.$$

The proof of (i) follows from the proof of 4.2.6. For the proof of (ii), let $S = c^{-1}\, ab^{\ell_1} \ldots c^{-1}\, ab^{\ell_{z-1}}$ and $T = c^{-1}\, ab^{\ell_z} \ldots c^{-1}\, ab^{\ell_m}$ where z is as in the proof of 4.2.8 from which the result follows.

□

4.2.11 Corollary: If $\beta \in B_n$, such that $\beta \equiv \bar{\beta}$ or $\beta \equiv F(\bar{\beta})$ and $\hat{\beta}$ is a knot then $\Phi(\beta) = u_1^o\, u_2^o \ldots u_k^o$ where $k = \frac{1}{2}(n - 1)$.

Proof: Note first that by results proved in chapter 3, n must be odd for such a β to exist. Let $m = n \cdot k = \binom{n}{2}$ and let $\gamma, \beta \in B_n$. Write β in its abelianized lifting form as in expression (3.2.1) where $\delta_{i,j}$ is the power of $g_{i,j}$. Denote by $L(\beta)$ the m-tuple $(\delta_{1,2}, \delta_{1,3}, \ldots, \delta_{n-1,n})$. $L(\beta)$ can be thought of as an element in the additive group G_m of m-tuples with integer entries, where the addition is defined coordinate-wise.

Recall from chapter 3 the natural homomorphism φ from B_n into S_n determined by the endpoints of the braids in B_n. Since φ is a homomorphism we have that for braids $\beta, \gamma \in B_n$, $e_{\beta^{-1}} = e_\beta^{-1}$ and $e_{\beta\gamma} = e_\gamma \circ e_\beta$.

Let S_n act on G_m as follows: For each $e \in S_n$ and $L \in G_m$ let

$$\begin{aligned} e \cdot L &= e \cdot (\delta_{1,2}, \delta_{1,3}, \ldots, \delta_{n-1,n}) \\ &= (\delta_{e(1),e(2)}, \delta_{e(1),e(3)}, \ldots, \delta_{e(n-1),e(n)}) \end{aligned}$$

For $\alpha,\beta,\gamma \in B_n$ we have the following

(4.2.12) $$e_\alpha \cdot L(\beta\cdot\gamma) = e_\alpha \cdot L(\beta) + e_\beta \circ e_\alpha \cdot L(\gamma)$$

Hence

$$0 = L(\beta^{-1}\cdot\beta) = L(\beta^{-1}) + e_{\beta^{-1}} \cdot L(\beta)$$

so that

$$L(\beta^{-1}) = - e_{\beta^{-1}} \cdot L(\beta).$$

Using these facts suppose that β,γ are such that $\beta = \gamma^{-1}\cdot\bar{\beta}\cdot\gamma$, where $\bar{\beta}$ is the obverse of β. Then

$$\begin{aligned} L(\beta) &= L(\gamma^{-1}\cdot\bar{\beta}\cdot\gamma) \\ &= L(\gamma^{-1}) + e_{\gamma^{-1}} \cdot L(\bar{\beta}\cdot\gamma) \\ &= L(\gamma^{-1}) + e_{\gamma^{-1}} \cdot L(\bar{\beta}) + e_{\bar{\beta}} \circ e_{\gamma^{-1}} \cdot L(\gamma) \end{aligned}$$

Since $e_{\gamma^{-1}} \cdot L(\bar{\beta}) = - e_{\gamma^{-1}} \cdot L(\beta)$ and $L(\gamma^{-1}) = - e_{\gamma^{-1}} \cdot L(\gamma)$ we have

$$0 = L(\beta) + e_{\gamma^{-1}} \cdot L(\beta) + e_{\gamma^{-1}} \cdot L(\gamma) - e_{\bar{\beta}} \circ e_{\gamma^{-1}} \cdot L(\gamma)$$

Pre-multiplying by e_γ yields

$$\begin{aligned} 0 &= e_\gamma \cdot L(\beta) + L(\beta) + L(\gamma) - e_\gamma \circ e_{\bar{\beta}} \circ e_{\gamma^{-1}} \cdot L(\gamma) \\ &= e_\gamma \cdot L(\beta) + L(\beta) + L(\gamma) - e_\beta \cdot L(\gamma) \end{aligned}$$

If we then let $L(\beta) = (\delta_{1,2}, \delta_{1,3}, \ldots, \delta_{n-1,n})$ and $L(\gamma) = (\zeta_{1,2}, \zeta_{1,3}, \ldots, \zeta_{n-1,n})$ then

(4.2.13) $$0 = \delta_{e_\gamma(i),e_\gamma(j)} + \delta_{i,j} + \zeta_{i,j} - \zeta_{e_\beta(i),e_\beta(j)}.$$

Since $\hat{B}$ is assumed to be a knot, $e_\beta{}^i(1)$ are unique for each $i = 1,2,\ldots,n$ with $e_\beta{}^n(1) = 1$. Also, using properties of the permutation group on n elements we have that if e_γ is such that

$e_\beta = e_{\gamma^{-1}} \circ e_\beta \circ e_\gamma$ then $e_\gamma = e_\beta{}^i$ for some $0 < i \leqslant n$. From (4.2.13) then

$$0 = \sum_{i=1}^{n} [\delta_{e_\gamma(e_\beta{}^i(1)), e_\gamma(e_\beta{}^{i+1}(1))} + \delta_{e_\beta{}^i(1), e_\beta{}^{i+1}(1)}]$$

$$+ \sum_{i=1}^{n} [\zeta_{e_\beta{}^i(1), e_\beta{}^{i+1}(1)} - \zeta_{e_\beta{}^{i+1}(1), e_\beta{}^{i+2}(1)}].$$

Because $e_\gamma = e_\beta{}^j$ for some j we have

$$0 = 2\sum_{i=1}^{n} \delta_{e_\beta{}^i(1), e_\beta{}^{i+1}(1)}.$$

Similarly, we have that for all $1 \leqslant j \leqslant n$,

$$0 = 2\sum_{i=1}^{n} \delta_{e_\beta{}^i(1), e_\beta{}^{i+j}(1)}.$$

Notice now that $\Phi(\beta)$ identifies $g_{1, e_\beta{}^j(1)}$ with $g_{e_\beta{}^i(1), e_\beta{}^{i+j}(1)}$ for each $i = 1, 2, \ldots, n$. Hence $\sum_{i=1}^{n} \delta_{e_\beta{}^i(1), e_\beta{}^{i+j}(1)} = 0$ are the powers of u_j for $1 \leqslant j \leqslant \frac{1}{2}(n-1)$ in $\Phi(\beta)$ which completes the proof. □

4.3 The Kauffman conjecture

For the remainder of chapter 4 we shall concern ourselves with a conjecture of Kauffman's [**K3**] and related matters. Henceforth, we assume that all braid diagrams yield closed braid diagrams that are alternating. Such a braid diagram, W will be called an alternating braid diagram and the braid it represents an alternating braid.

Notice that the sign of all crossings in any one column of an alternating braid diagram are the same and must be opposite to the sign in any adjacent column.

For any braid diagram W let $sh(\hat{W})$ denote the link diagram, $\hat{W}$ with a checkerboard shading as in [**K3**,p.22] and let $U(\hat{W})$ be the number of unshaded regions. (The set of unshaded regions includes the unbounded region). Likewise, let $V(\hat{W})$ be the number of shaded regions. Also let $\rho_W(i)$ be the sum of the signs of the crossings in column i, of W. Then we have

(4.3.1)(1) For n odd, $U(\hat{W}) = |\Sigma\, \rho_W(k)| + 1$ if k is even and $V(\hat{W}) = |\Sigma\, \rho_W(k)| + 1$ if k is odd.

(2) For n even, $U(\hat{W}) = |\Sigma\, \rho_W(k)| + 2$ if k is even and $V(\hat{W}) = |\Sigma\, \rho_W(k)|$ if k is odd.

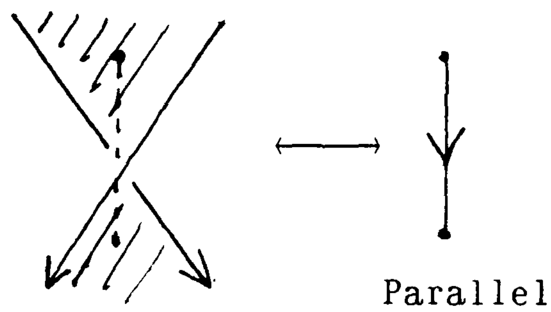
Hence

(3) $\omega(\hat{W}) = 0 \iff |U(\hat{W}) - V(\hat{W})| = 0$ for n odd and

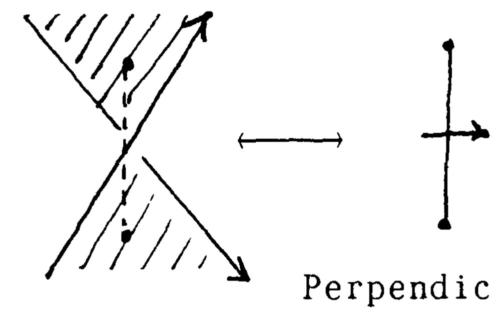
(4) $\omega(\hat{W}) = 0 \iff |U(\hat{W}) - V(\hat{W})| = 2$ for n even.

For a link diagram A one defines the planar graph $G(A)$ as in [**K4**] as follows: the vertices of $G(A)$ are in one-to-one correspondence with the shaded regions of $sh(A)$. For each crossing in the diagram in which two regions touch, the corresponding vertices are joined by an edge in the plane. Placed on each edge, e, is an orientation, which we say is either <u>parallel to e</u> or <u>perpendicular</u>

to e, depending on the orientation of the link diagram (see figure below).



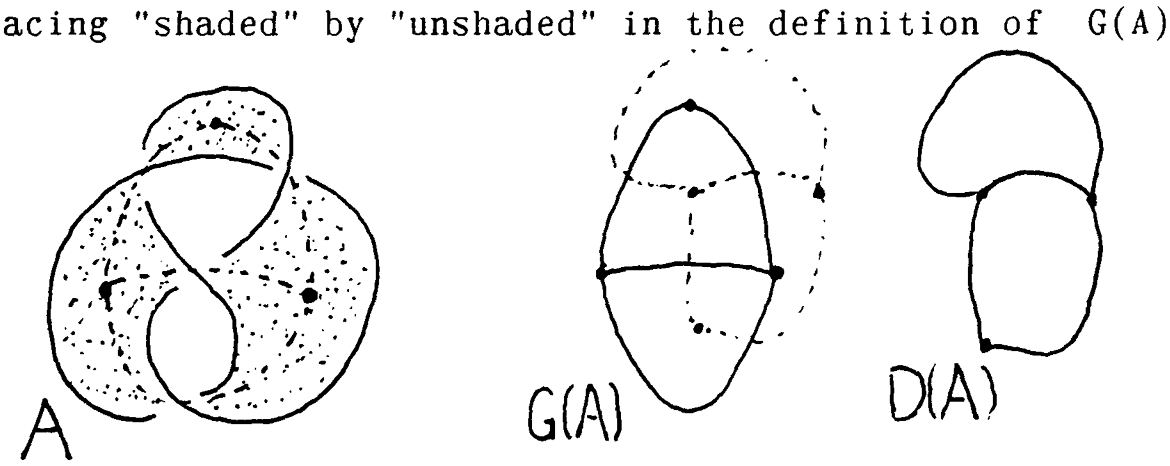
Parallel
to e



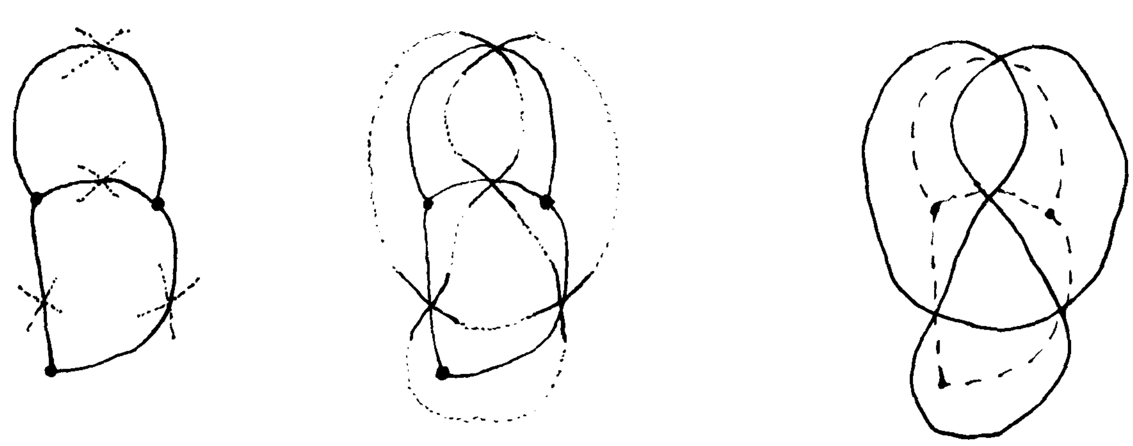
Perpendicular
to e

Call the graph an oriented graph if each edge has an orientation. Unless otherwise stated the orientations on the edges of the graph will be ignored.

The dual graph, D(A), is the planar graph obtained by taking the planar dual of G(A) and is identical to the graph found by replacing "shaded" by "unshaded" in the definition of G(A).



Also, for every graph G there is an induced unoriented link universe found by associating a double point to each edge of G. Two points are joined by an arc if their corresponding edges are incident at a vertex of G as shown by the following figure.

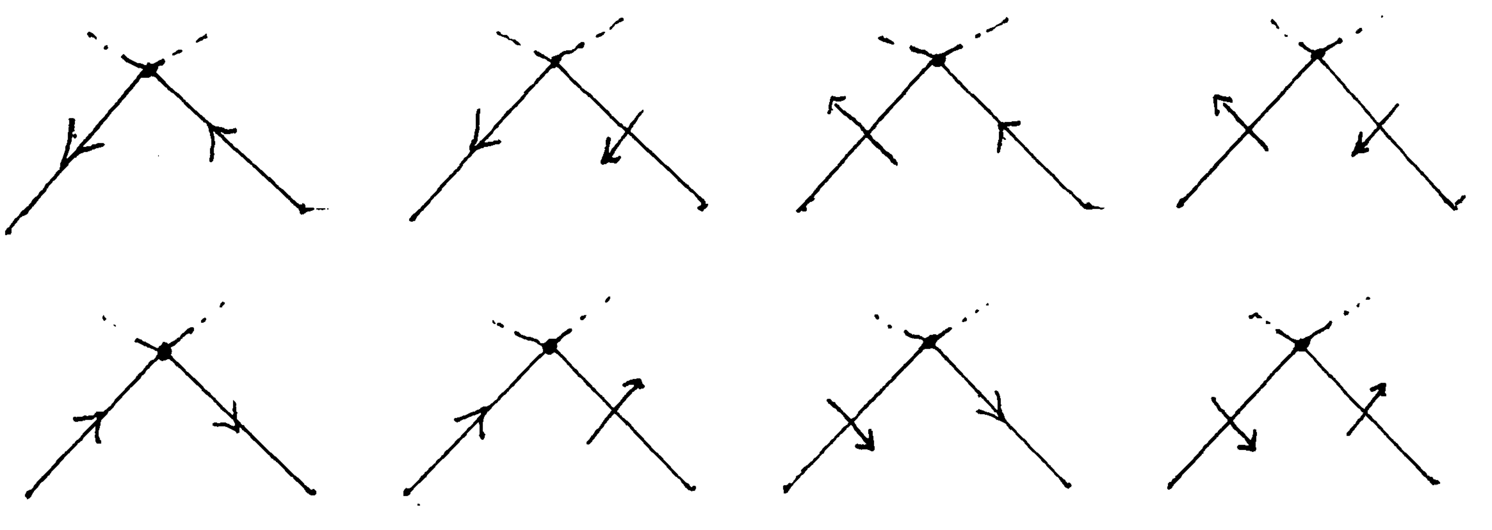


(4.3.2) Notes: (1) Every (unoriented) link universe has two associated (unoriented) alternating link diagrams; one being the obverse of the other.

(2) If $G(A)$ and $H(A')$ are two oriented graphs obtained from link diagrams A and A' that differ only by the orientation on their components and if any edge of $G(A)$ has a parallel orientation while the orientation on the corresponding edge in $H(A')$ is perpendicular then $c(A) = c(A') > 1$.

(3) In this section all graphs will be assumed to be planar.

(4) If G is an oriented graph then clearly G induces an oriented link universe if and only if every two edges that meet at a vertex and are adjacent have one of the eight following orientations.



We now abstractly define a "braid graph" and note that every closed braid diagram satisfies the definition . However, the converse, that every braid graph defines a closed braid diagram, is not true.

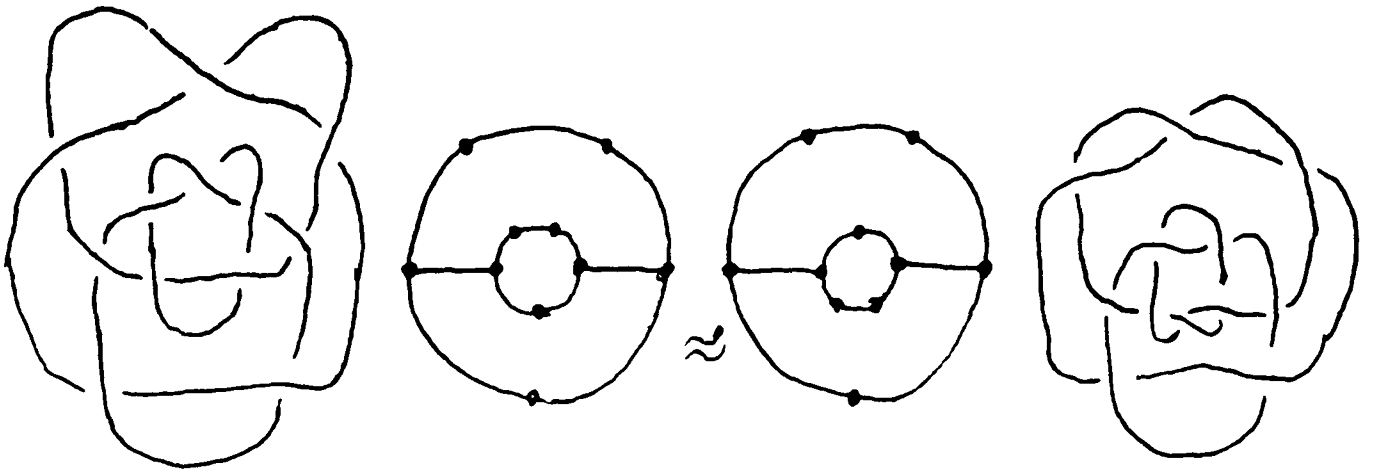
<u>4.3.3 Definition</u>: An oriented graph G with vertices V and edges E is said to be a <u>braid graph</u> if there exists (1) a partition of $V = \cup V_j$ such that for each $V_j = \{v_{j_1}, v_{j_2}, \ldots, v_{j_k}\}$ there exists a single edge with parallel orientation between v_{j_h}, v_{j_i} if and only if $h \equiv i \pm 1$ (modulo k) (Call V_j with the oriented edges

between its vertices the j-th cycle of G, C_j) and (2) an edge of G with perpendicular orientation between C_i and C_j only if $i = j \pm 1$ (up to a reordering of the indices).

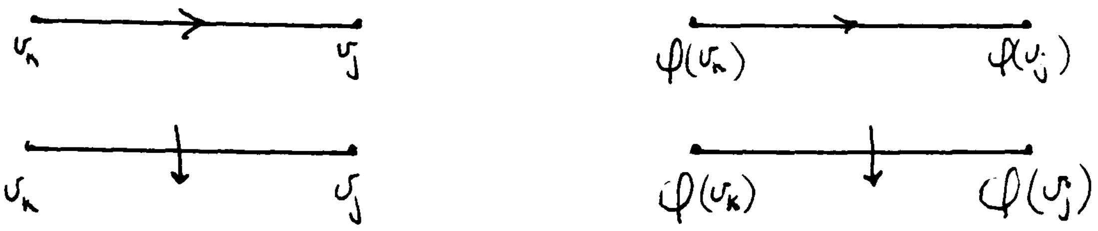
4.3.4 Definition: Let G and H be graphs. A map $\varphi : G \longrightarrow H$ is said to be a graph isomorphism if φ is a bijection between both edges and vertices respecting incidences. We write $G \approx H$.

Remark: The existence of an isomorphism between the graphs of two reduced alternating diagrams of knots is a strong condition, which one might expect to be sufficient for the knots to be isotopic, possibly after changing all the signs of the crossings of one of the knots. Such a conjecture would not extend to links. Example 1. below demonstrates two link diagrams with isomorphic graphs where the corresponding links are not isotopic (nor can be made isotopic by switching all the signs of one of the links).

Example 1.



We note that if the definition were extended to include the orientations on the edges (as illustrated below) then $\varphi(G(\hat{W}))$ for some braid words W is a braid graph but need not be the graph of an oriented link diagram as the above example also demonstrates.



We now resrict our attention to closed braid diagrams that are not only alternating but are knots diagrams as well.

In [**K3**,p.32], Kauffman conjectures that if a reduced alternating diagram A represents an [unorientedly] achiral knot, then $|U(A) - V(A)| = 0$ and the graph, G(A), obtained from the shading of A will be isomorphic to its dual, D(A).

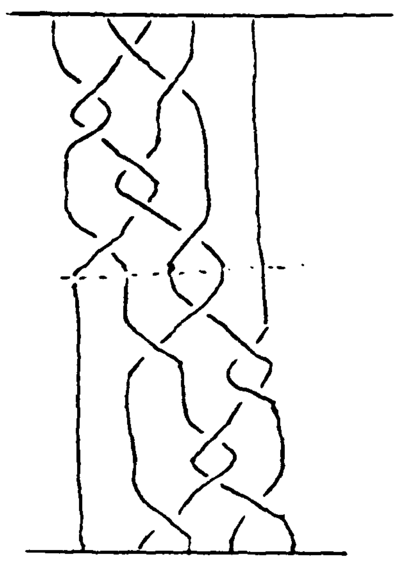
Recall that by corollary 3.3.3 the writhe of a 1-component closed braid diagram is equal to one less than the number of strings modulo 2. Using the Tait conjecture, now proved by Thistlethwaite, [T] , the writhe of a reduced alternating diagram of an achiral knot must be zero and hence in Kauffman's conjecture we need only consider braid diagrams of odd string number. Immediately we have $|U(A) - V(A)| = 0$.

However, using flypes on visibly achiral and visibly reversed achiral braid diagrams we construct examples of two five-stringed closed reduced alternating braid diagrams whose graphs are not isotopic to their duals (see examples 2. and 3.). Furthermore, by "weaving in" (see example 4.) another 2m strings for any positive integer m, we may also construct counter-examples in B_n for all odd $n > 3$.

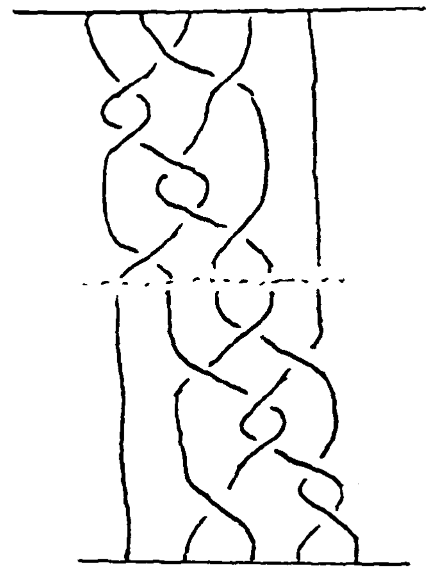
<u>Example 2.</u> Let $W = V \cdot R(\bar{V})$ where $V = \sigma_2^{-1} \sigma_3 \sigma_1^{2} \sigma_2^{-2} \sigma_3 \sigma_1$. Using a flype as in section 3.1 we have that $\hat{W}$ is isotopic to $\hat{W}'$ where $W' = \sigma_2^{-1} \sigma_3 \sigma_1^{2} \sigma_2^{-2} \sigma_3 \sigma_1 . \sigma_3 \sigma_2^{-1} \sigma_4^{-1} \sigma_3^{2} \sigma_2^{-1} \sigma_4^{-2}$. Notice that $\hat{W}'$

is a reduced alternating knot diagram.

W

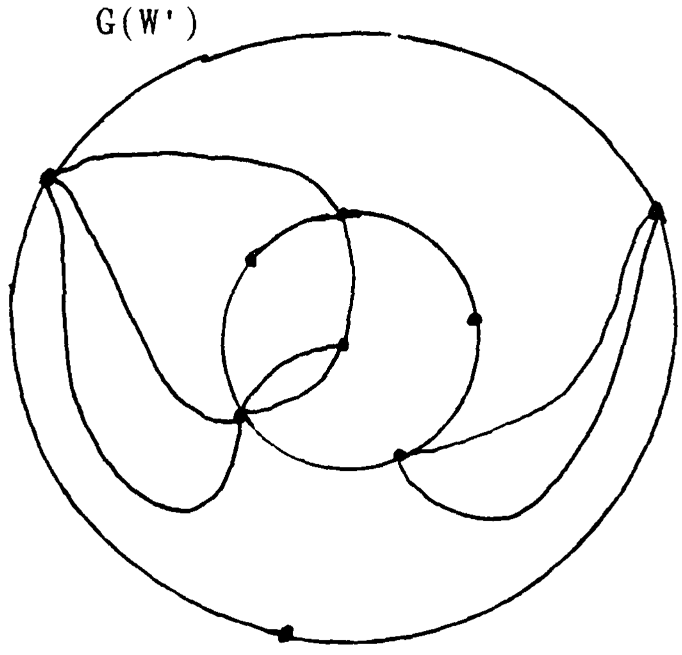


W'

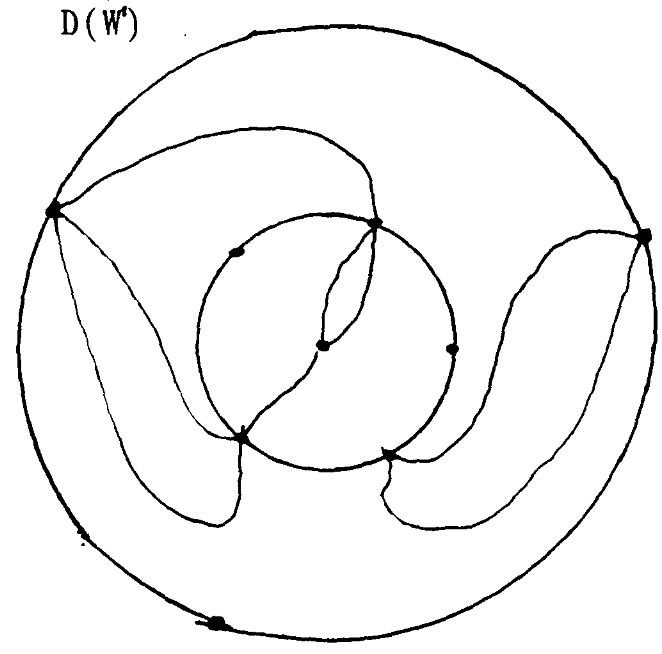


Form the graph, $G(\hat{W}')$, and dual graph, $D(\hat{W}')$, of $\hat{W}'$. In $G(\hat{W}')$ there exists a vertex at which six edges are incident. This is not true for $D(\hat{W}')$, hence, there does not exist an isomorphism between the two.

$G(\hat{W}')$

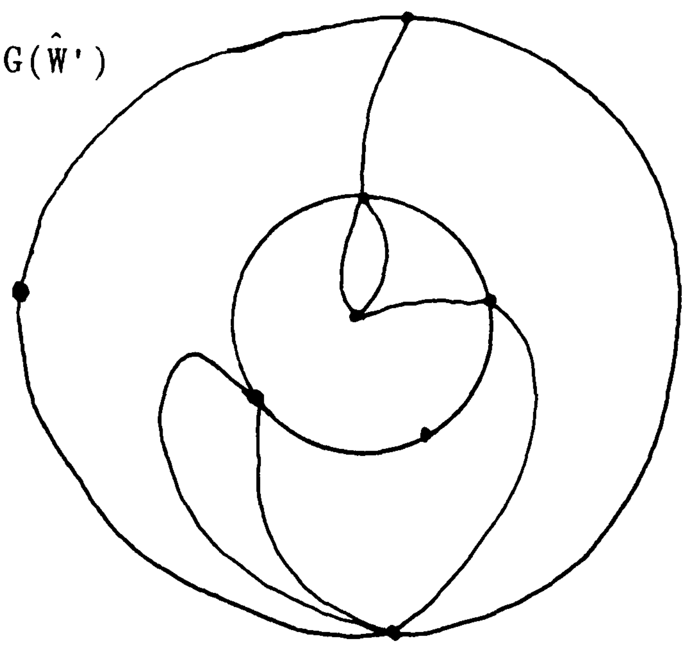


$D(\hat{W}')$

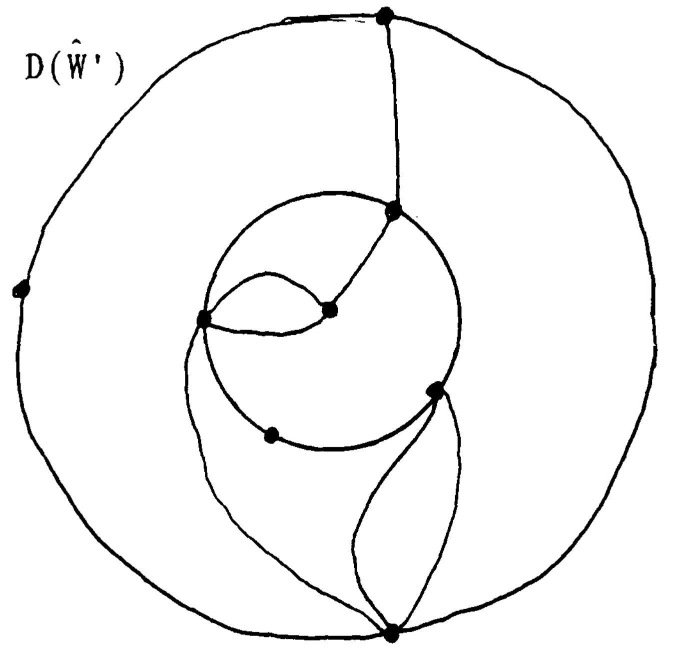


<u>Example 3</u>. Let $W = V \cdot F \circ R(\bar{V})$ where $V = \sigma_1 \sigma_2^{-1} \sigma_1^{2} \sigma_3 \sigma_2^{-2}$ then $W' = \sigma_1 \sigma_2^{-1} \sigma_1^{2} \sigma_3 \sigma_2^{-2} \cdot \sigma_3^{2} \sigma_2^{-1} \sigma_4^{-1} \sigma_3 \sigma_4^{-2}$

$G(\hat{W}')$



$D(\hat{W}')$



Example 4. Given a reduced alternating braid word W on n strings whose closure is a knot, we can construct a new word, called the weave of W, with the same properties, on $n + 2$ strings by the following. Let $a = \frac{1}{2}(n - 1)$ and $\varepsilon, \delta = \pm 1$, then define

ψ : {Diagrams on n strings} $\longrightarrow$ {Diagrams on n+2 strings} by

(i) $\sigma_i^{\varepsilon} \mapsto \sigma_i^{\varepsilon}$, for $1 \leqslant i < a$

(ii) $\sigma_a^{\delta} \mapsto \sigma_{a+1}^{-\delta}\ \sigma_a^{\delta}\ \sigma_{a+1}^{-\delta}$

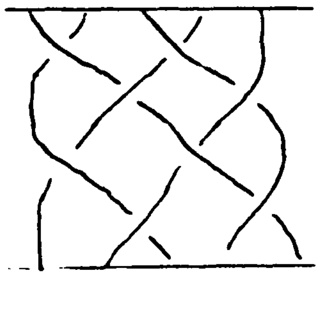
(iii) $\sigma_{a+1}^{-\delta} \mapsto \sigma_{a+2}^{\delta}\ \sigma_{a+3}^{-\delta}\ \sigma_{a+2}^{\delta}$

(iv) $\sigma_i^{\varepsilon} \mapsto \sigma_{i+2}^{\varepsilon}$, for $a + 1 < i \leqslant n - 1$.

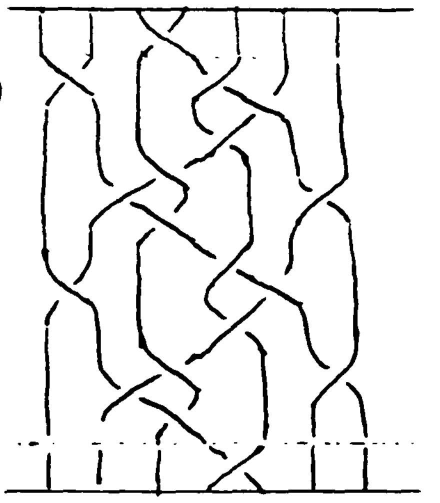
Thus, define the weave of W by

$$\text{Weave}(W) = \sigma_{a+1}^{-\delta} \cdot \psi(W) \cdot \sigma_{a+2}^{\delta} .$$

W



Weave(W)



The construction ensures that the Weave(W) is visibly reversed achiral if and only if W is visibly reversed achiral (a similar construction would give the same result for visibly achiral diagrams).

Pending the validity of the conjecture remarked on at the end of section 4.2, along with the corollaries 4.2.9 and 4.2.10, it would be true that every achiral knot with a three braid representative has an alternating closed braid diagram. Furthermore, every alternating 3-string closed braid diagram of an achiral knot would be of the form $(V\cdot R(\bar{V}))^m$ or $(V\cdot F \circ R(\bar{V}))^m$. Noting that the graphs of such diagrams are clearly isomorphic to their duals, Kauffman's conjecture would be true for alternating closed braid diagrams on 3-strings. Note that we have deliberatly left out the word "reduced" since non-reduced alternating braid diagrams on 3-string must be reducible braid words, i.e. of the form $\sigma_1^{\ k}\ \sigma_2^{\ \pm 1}$, up to conjugation. The only braid words of this form representing achiral knots are $\sigma_1\ \sigma_2^{\ -1}$ or $\sigma_1^{\ -1}\ \sigma_2$ whose graphs are isomorphic to their duals.

We now consider a converse to Kauffman's conjecture.

<u>4.3.5 Theorem</u>: Let the knot diagram, $\hat{W}$, be a reduced alternating closed braid diagram representing an alternating braid, $\beta \in B_n$, then if $G(\hat{W}) \approx D(\hat{W})$, β is (visibly) unorientedly achiral.

<u>Proof</u>: Assume $\beta \in B_n$ and W are as in the theroem. We first point out that for such an isomorphism to exist implies $U(\hat{W}) = V(\hat{W})$ and by 4.3.1 $|\omega(\hat{W})| = 0$ or 2 accordingly as n is odd or even. By corollary 3.3.4, however, the writhe of a knot $\hat{W}$ on an even

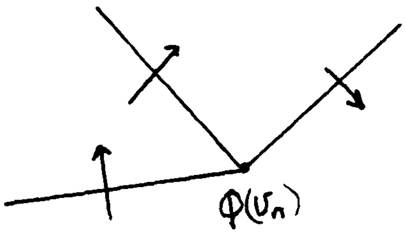
number of strings must be odd. Hence, as before, we need only be concerned with W on an odd number of strings. Recall that with $W \in B_n$ for n odd, the region containing the axis point is shaded and the vertex of $G(\hat{W})$ that corresponds to this region is denoted by v_n. Likewise, the vertex of $D(\hat{W})$ associated to the unbounded region is denoted by v_0.

The proof of the theorem hinges on the fact that the isomorphism sends v_n to v_0 as well as sending cycles in $G(\hat{W})$ to cycles in $D(\hat{W})$, as the following two lemmas show.

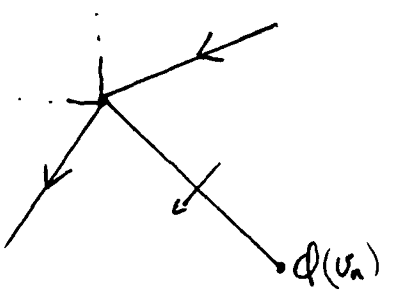
<u>4.3.6 Lemma</u>: Let $G(\hat{W})$ and $H(\hat{U})$ be the graphs obtained from the closed braid diagrams of odd string number $\hat{W}$, $\hat{U}$, with $V = \cup V_j$, $j = 1,3,\ldots,n$, a partition of the vertices of $G(\hat{W})$. Let $\varphi : G(\hat{W}) \longrightarrow H(\hat{U})$ be a graph isomorphism. Then there exists an orientation on the components of $\hat{U}$ such that $\varphi(G(\hat{W}))$ is a braid graph using $\varphi(V) = \cup\, \varphi(V_j)$ as its partition.

<u>Proof of lemma</u>: If e is the single edge of the cycle C_j with endpoints v_{ji}, and v_{ji+1} then certainly $\varphi(e_{ji})$ is a single edge between $\varphi(v_{ji})$ and $\varphi(v_{ji+1})$. Likewise, if e is an edge between C_j and C_{j+1} then $\varphi(e)$ is an edge between $\varphi(C_j)$ and $\varphi(C_{j+1})$. Hence we need only show that , for some orientation on $\hat{U}$, the edges of $\varphi(C_j)$ have parallel orientations and those between $\varphi(C_i)$ and $\varphi(C_j)$, $i \neq j$, have perpendicular orientations. We prove this by constructing such an orientation.

Start with the point $\varphi(v_n)$ in $H(\hat{U})$. For each of the $\rho_W(n-1)$ edges incident at $\varphi(v_n)$ construct a perpendicular orientation as in the figure below.



For the edges of $\varphi(C_{n-2})$ that share an endpoint with one of the edges incident at $\varphi(v_n)$, place on the edge a parallel orientation as shown here.



This then allows a parallel orientation on all edges of $\varphi(C_{n-2})$ and so $\varphi(C_{n-2})$ becomes a cycle. In general each cycle, $\varphi(C_i)$, induces a perpendicular orientation on the edges that have $\varphi(C_i)$ and $\varphi(C_{i-2})$ as endpoints, which in turn induce a parallel orientation on $\varphi(C_{i-2})$. Hence, $\varphi(C_j)$, $1 \leqslant j \leqslant n - 2$, can be made into cycles. Using note (4.3.2)(3) we see that since each pair of adjacent edges with common endpoint is one of the eight allowable orientations and so $\varphi(G(\hat{W}))$ with the constructed orientation admits a link universe. This completes the proof of the lemma. □

<u>4.3.7 Lemma</u>: Let W be a braid word in B_n, for n odd. If (a) $\hat{W}$ is a knot, (b) $\varphi : G(\hat{W}) \longrightarrow D(\hat{W})$ is an isomorphism between the graph of $\hat{W}$ and its dual and (c) C_i for $i = 1,3,\ldots,n - 2$ and D_j for $j = n - 1,n - 3,\ldots,2$ are the cycles of $G(\hat{W})$ and $D(\hat{W})$, repectively, then $\varphi(C_i) = D_{n-i}$.

Proof of lemma: Suppose first that v_n in $G(\hat{W})$ is not mapped by φ to v_o in $D(\hat{W})$. That is, $\varphi(v_n)$ is some vertex in D_j for some j. By the proof of lemma 4.3.6 there exists an orientation on $\hat{W}$ such that all edges incident at $\varphi(v_n)$ have perpendicular orientation, and since $\varphi(v_n)$ is in D_j, $\hat{W}$ has an orientation such that at least one edge incident at $\varphi(v_n)$ is of parallel orientation, contradicting the fact that $\hat{W}$ is a knot. Hence,

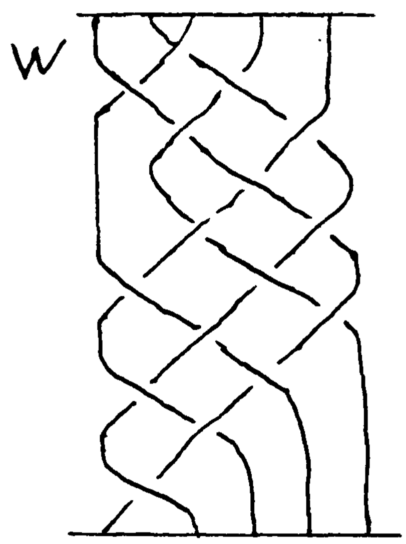
(4.3.8) $$\varphi(v_n) = v_o.$$

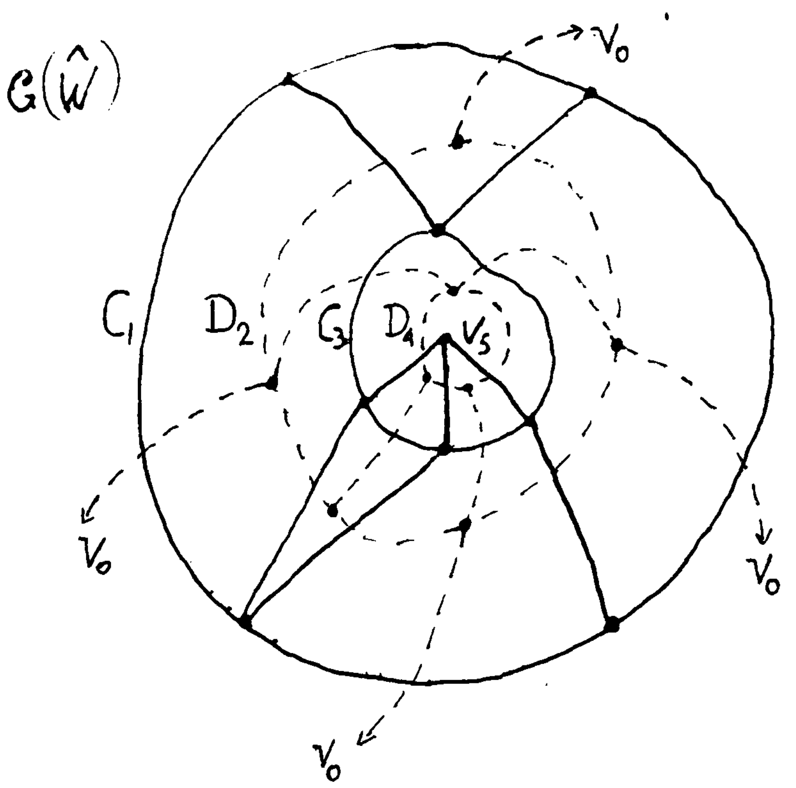
Suppose now for some k and for all $j < k$, j and k odd, that $\varphi(C_j) = D_{n-j}$ and that $\varphi(C_k) \neq D_{n-k}$. Since $\hat{W}$ is a knot there exists an edge e of $G(\hat{W})$ with endpoints in C_k and C_{k-2}. By the induction $\varphi(e)$ has an endpoint in D_{n-k+2}, and not in $\varphi(C_{k-4}) = D_{n-k+4}$, and since $D(\hat{W})$ is a braid graph, $\varphi(e)$ must have its other endpoint in D_{n-k}. That is, at least one vertex of C_k is mapped to D_{n-k}. By the supposition, there exists an edge f of C_k that is not mapped to D_{n-k}, i.e. $\varphi(f)$ has endpoints in D_{n-k} and D_{n-k-2}, so that by the proof of lemma 4.3.6 there exist orientations on $\hat{W}$ that yield both parallel and perpendicular orientations on f. This is a contradiction since $\hat{W}$ is a knot. Hence

(4.3.9) $$\varphi(C_i) = D_{n-i}$$

for all $i = 1,3,\ldots,n-2$. This completes the proof of the lemma. □

Remark: The equivalent statement is not true for braid diagrams whose closures are links of more than one component. The following is an example of a closed braid with graph $G(\hat{W})$ and dual $D(\hat{W})$ such that the isomorphism between them takes v_5 to a vertex in D_2.





To establish achirality and the theorem we employ the map $r_{s,t}$: {Diagrams on n strings} $\longrightarrow$ {Diagrams on 3 strings} for s,t such that $|s - t| = 1$ defined by $\sigma_s \mapsto \sigma_1$,$\sigma_t \mapsto \sigma_2$ and $\sigma_i \mapsto 1$ for $i \neq s,t$. When clear from the context we write $r_{s,t}$ when we mean $r_{s,t}(\hat{W})$ for a braid diagram W.

We use lemma 4.3.7 as a starting point. Not only must the edges incident at v_n of $G(\hat{W})$ be in one-to-one correspondence with the edges incident at v_0 in $D(\hat{W})$, but their relation with C_{n-2} and D_2, respectively, allow the following statement; if

$$r_{1,2} \equiv \sigma_1^{\beta_1}\ \sigma_2^{-\beta_2}\ \sigma_1^{\beta_3}\ \sigma_2^{-\beta_4} \ldots \sigma_1^{\beta_{k-1}}\ {}^{\beta_k}$$

and

$$r_{n-1,n-2} \equiv \sigma_1^{-\gamma_1}\ \sigma_2^{\gamma_2}\ \sigma_1^{-\gamma_3}\ \sigma_2^{\gamma_4} \ldots \sigma_1^{-\gamma_{h-1}}\ \sigma_2^{\gamma_h}$$

where β_i, γ_i are either all negative or all positive integers, then the sequence of integers $(\beta_1, \beta_2, \ldots, \beta_k)$ must be a cyclic permutation of $(\gamma_1, \gamma_2, \ldots, \gamma_h)$. That is, $k = h$ and either $\beta_i = \gamma_{x+i}$ or $\beta_i = \gamma_{x-i}$ for all $i \in \mathbb{Z}$ and some $x \in 2\mathbb{Z}$, where subscripts are considered as elements in $\mathbb{Z}_k$.

In general, the sequence corresponding to the powers of $r_{i,i+1}$ must be a cyclic permutation of the sequence corresponding to the powers of $r_{n-i,n-i-1}$. Specifically, if we let $i = a = \frac{1}{2}(n-1)$ we then have

$$r_{a,a+1} \equiv \sigma_1^{\alpha_1}\ \sigma_2^{-\alpha_2}\ \sigma_1^{\alpha_3}\ \sigma_2^{-\alpha_4} \ldots \sigma_1^{\alpha_{2m-1}}\ \sigma_2^{-\alpha_{2m}}$$

for some $m \in \mathbb{Z}$, where α_i are either all negative or all positive, and that

$$r_{n-a,n-a-1} = r_{a+1,a} \equiv \sigma_2^{\alpha_1}\ \sigma_1^{-\alpha_2}\ \sigma_2^{\alpha_3}\ \sigma_1^{-\alpha_4} \ldots \sigma_2^{\alpha_{2m-1}}\ \sigma_1^{-\alpha_{2m}}$$

so that $\alpha_m = -\alpha_x$ for some odd x, $1 \leqslant x \leqslant 2m$ and either (a) $\alpha_i = -\alpha_{x+i}$ for all $i \in \mathbb{Z}_{2m}$ or (b) $\alpha_i = -\alpha_{x-i}$ for all $i \in \mathbb{Z}_{2m}$.

Case (a) This implies that $\alpha_i = \alpha_{2x+i}$ for all $i \in \mathbb{Z}_{2m}$ so that $\alpha_i = \alpha_{d+i}$ where $d = \mathrm{GCD}(2x, 2m)$. Let $cd = 2m$. Since x odd, $\frac{1}{2} d$ is an odd integer. Hence, for $e = \frac{1}{2} d$

$$\begin{aligned} r_{a,a+1} &\equiv (\sigma_1^{\alpha_1}\ \sigma_2^{-\alpha_2} \ldots \sigma_1^{\alpha_e}\ \sigma_2^{-\alpha_1} \ldots \sigma_2^{-\alpha_e})^c \\ &\equiv (S\cdot \mathcal{R}(\bar{S}))^c \qquad \text{, for some braid word } S \in B_3. \end{aligned}$$

Case (b) We have that if $b = \frac{1}{2}(x-1)$ and $f = \frac{1}{2} m$ then $\alpha_b = -\alpha_{b+1}$ and $\alpha_{b+f} = \alpha_{b+f+1}$. Hence,

$$\begin{aligned} r_{a,a+1} &\equiv \sigma_1^{\alpha_{b+1}}\ \sigma_2^{-\alpha_{b+2}} \ldots \sigma_{j_i}^{\varepsilon\alpha_{b+f}}\ \sigma_{j_{i+1}}^{-\varepsilon\alpha_{b+f}} \ldots \sigma_1^{\alpha_{b+2}}\ \sigma_2^{-\alpha_{b+1}} \\ &\equiv T\cdot \mathcal{F} \circ \mathcal{R}(\bar{T}) \qquad \text{, for some braid word } T \in B_3, \end{aligned}$$

where $j_i = 1$ or 2 and $\varepsilon = +1$ or -1 accordingly as $\frac{1}{2} m$ is odd or even and $j_{i+1} = -j_i$.

It is not hard to show that this structure on the centre two columns of W forces a similar structure on adjacent columns, hence, throughout W. We conclude that W is visibly unorientedly achiral and thus is unorientedly achiral. □

Appendix A

A reversing result for 3-component links

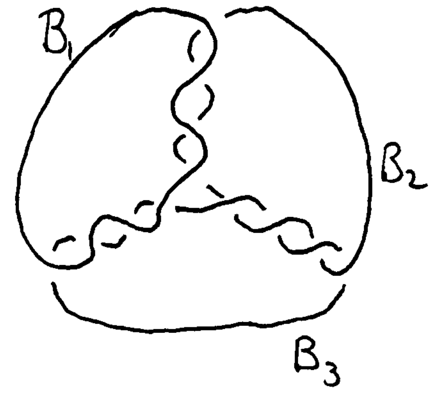
As with propositions 2.1.11 and 2.2.5 our aim is to consider the effect different orientations on the components of a 3-component link have on τ_3. The size and complexity of this result seem to highlight the difference between the Jones polynomial and its two variable generalization. We give the result in hopes that patterns might be seen or that it may help in testing future reversing conjectures for $P(A)(\ell,m)$.

Let A be a 3-component link diagram. Let the knot diagrams formed by the components, separately, be B_1, B_2 and B_3. Denote by A_i the two component link diagram obtained from A by removing the i-th component so that $A = A_i \cup B_i$. Notice that by P3. reversing the orientations of both B_1 and B_2 affects the Conway polynomial, and hence τ_3, in the same way as reversing only B_3. Therefore, to see the affects on τ_3 of any orientation change we need only consider changing the orientation of B_3 up to a reordering of the components. Then let $\tilde{B}_i$ be the knot diagram with its orientation reversed and write $\tilde{A} = A_3 \cup \tilde{B}_3$. We have then

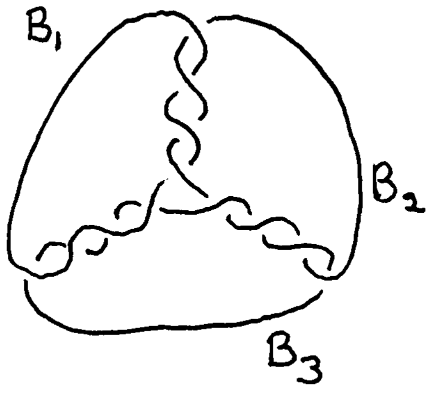
Proposition: Let A, A_i, B_i, $\tilde{A}$ and $\tilde{B}_i$ be the link diagrams defined above. Let $a = \lambda(B_1, B_2)$, $b = \lambda(B_1, B_3)$ and $c = \lambda(B_2, B_3)$, then

$$\begin{aligned}\tfrac{1}{2}\cdot[\tau_3(A) - \tau_3(\tilde{A})] &= b\cdot\tau_2(A_1) + c\cdot\tau_2(A_2) + (b + c)\cdot\tau_2(A_3)\\ &\quad + a(b + c)\cdot\tau(B_3) - bc\cdot[\tau(B_1) + \tau(B_2) + 2\cdot\tau(B_3)]\\ &\quad + \frac{1}{12}[b^3c - 2bc + bc^3 + a(b^3 + 3b^2c - b - c + 3bc^2 + c^3)]\end{aligned}$$

<u>Proof</u>: We prove 2.4.1 in stages, each assuming more complicated linking than the stage before. Case (a) assumes one component is separated from the rest. Case (b) assumes B_3 is unlinked with B_1, yet linked with B_2. Case (c) has only B_1 and B_2 unlinked, and case (d) is the general case where nothing is assumed about the linking.



Case (a) Assume $A \cong A_1 \sqcup B_1$. By the definition $\tau_3(A) = \tau_3(\tilde{A}) = \tau_2(A_2) = \tau_2(A_3) = a = b = 0$ and the result follows.



Case (b) Here assume $A_2 \cong B_1 \sqcup B_3$ so that $\tau_2(A_2) = b = 0$. Let ξ_j, $j = 1,2,\ldots,s$ be a sequence of crossing switches, from sign ε_j to $-\varepsilon_j$, that unlink B_1 and B_2. Then

$$\tau_3(A) = \sum_{j=1}^{s} \varepsilon_j \cdot \tau_2(C_j)$$

where $C_j = \eta_j \prod_{i<j} \xi_i A$. Let $\tilde{C}_j = \eta_j \prod_{i<j} \xi_i \tilde{A}$, then by 2.2.5

$$\tau_2(C_j) = \tau_2(\tilde{C}_j) + 2c[\tau(\eta_j \prod_{i<j} \xi_i A_3) + \tau(B_3)] + \frac{1}{6}(c^3 - c)$$

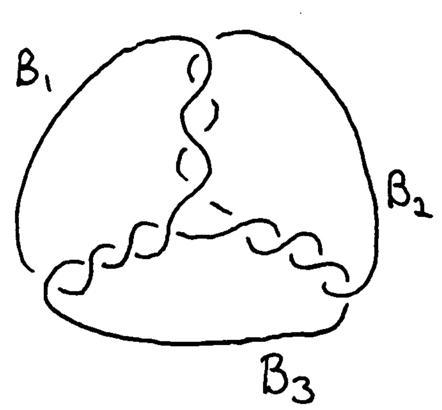
<u>Notes</u>: (1) $\sum_j \varepsilon_j \cdot \tau_2(\tilde{C}_j) = \tau_3(\tilde{A})$

(2) $\sum_j \varepsilon_j \cdot \tau(\eta_j \prod_{i<j} \xi_i A_3) = \tau_2(A_3)$

(3) $\sum_j \varepsilon_j = a$

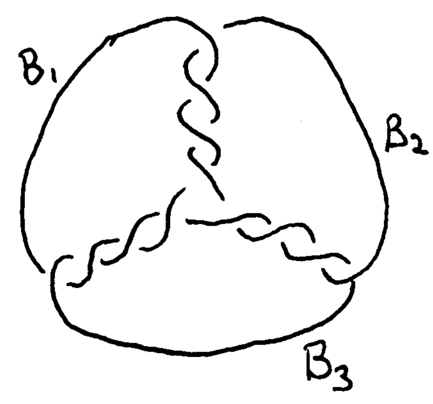
Thus

$$\tau_3(A) = \tau_3(\tilde{A}) + 2c\cdot\tau_2(A_3) + 2ac\cdot\tau(B_3) + \frac{a}{6}(c^3 - c)$$



Case (c) Take A such that $A_3 \cong B_1 \sqcup B_2$. Using case (b) we can find the relation between $\tau_3(A_2 \cup B_2)$ and $_3(A_2 \cup \tilde{B}_2)$, and between $\tau_3(A_2 \cup \tilde{B}_2) = \tau_3(B_1 \cup \tilde{B}_2 \cup B_3)$ and $\tau_3(\tilde{B}_1 \cup \tilde{B}_2 \cup B_3) = \tau_3(A)$. Note that $\tau_2(A_3) = a = 0$ and that $\lambda(\tilde{B}_2, B_3) = -c$. Then

$$(*) \qquad \begin{aligned} \tau_3(A) &= \tau_3(\tilde{A}) + 2b\cdot\tau_2(A_1) + 2c\cdot\tau_2(A_2) \\ &\quad - 2bc[\tau(B_1) + \tau(B_2) + 2\cdot\tau(B_3)] \\ &\quad + \frac{1}{6}(b^3c - 2bc + bc^3) \end{aligned}$$



Case (d) For any 3-component link diagram A, let the sequence of crossing switches, ξ_i, $i = 1,2,\ldots,r$, unlink the components B_1 and B_2. The result is a diagram, A', that satisfies the conditions of case (c). Hence,

$$\tau_3(A) = \sum_j \varepsilon_j \cdot \tau_2(C_j) + \tau_3(A')$$

where $C_j = \eta_j \prod_{i<j} \xi_i(A_3 \cup B_3)$. Using 2.2.5 once more on C_j we have that

$$\tau_2(C_j) = \tau_2(\tilde{C}_j) + 2(b + c)[\tau(\eta_j \prod_{i<j} \xi_i A_3) + \tau(B_3)] + \frac{1}{6}[(b + c)^3 - (b + c)]$$

Notes: Here we refer to the notes above replacing (1) by

(1') $\quad \sum_j \varepsilon_j \cdot \tau_2(\tilde{C}_j) + \tau_3(\tilde{A}') = \tau_3(\tilde{A})$

Thus

$$\tau_3(A) = \sum_j \varepsilon_j \cdot \tau_2(\tilde{C}_j) + \tau_3(A') + 2(b + c)\cdot\tau_2(A_3) + 2a(b + c)\cdot\tau(B_3) + \frac{a}{6}[(b + c)^3 - (b + c)]$$

Substituting the right-hand-side of (*) into $\tau_3(A')$ of (**) and applying note (1') gives the desired result. □

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